Hydraulic Estimation of Dispersion Coefficient for Streams

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Abstract: Inclusion of longitudinal dispersion enhances realism in the assessment of transport in streams by accounting for contaminant mixing. Here we revisit the problem of estimating the longitudinal dispersion coefficient D from readily measurable bulk hydraulic variables, which is useful in applications. We develop and calibrate, through field testing, a predictor for the longitudinal dispersion coefficient for streams that is structured after the fundamental formula of G. I. Taylor. Comparison of predicted with observed values of D shows the new optimized formula providing closer estimates than the established formula of Fischer, with respect to both mean accuracy and deviations from the mean, achieving in 14 of the 24 cases studied predictive accuracy of $0.5 \le D_{\text{observed}}/D_{\text{predicted}} \le 2$.

INTRODUCTION

Prediction of water quality in streams requires simultaneous consideration of mass transport and chemical reactions. In this contribution, we are concerned with transport in a one-dimensional framework in which the two major operative processes are advection and disperson. Though in environmental engineering analyses longitudinal dispersion is often ignored, its inclusion enhances accuracy in the assessment of transport by accounting for contaminant mixing, and it is essential whenever substantial longitudinal concentration gradients exist; this is likely when the stream loading is rapid. The mass balance for a conservative solute, stated in terms of the mean cross-sectional concentration C, is written in the form

$$\frac{\partial C}{\partial t} + U\frac{\partial C}{\partial x} = \frac{1}{A}\frac{\partial}{\partial x}\left(AD\frac{\partial C}{\partial x}\right) \cong D\frac{\partial^2 C}{\partial x^2} \quad (1)$$

in which U = the mean flow velocity in the cross section of area A; D = the longitudinal dispersion coefficient; x = the distance along the thalweg; and t = time (Holley 1967; Holley and Jirka 1986). Estimation of D from readily obtainable bulk hydraulic variables is useful in applications; here, we revisit this problem, investigating means to improve estimation accuracy.

In this work, we assume that the mass of a constituent that was introduced into the stream has mixed sufficiently with the ambient water and is occupying the entire cross section; meanwhile, its initial momentum difference has been reduced to the point that the solute has become hydrodynamically passive (a tracer). It is further assumed that the initial period has lapsed, after which the skewed longitudinal concentration distribution (relative to the position of the mean flow, the tracer spreads more upstream than downstream) begins to decay to a Gaussian one. Under these conditions, G. I. Taylor's (1953, 1954) shear flow dispersion theory is applicable, which states that the longitudinal spreading of mass depends on the cross sectional velocity distribution and on diffusion. Taylor derived this elegant and apparently simple result through incisive physical insight that achieved mathematical tractability.

Two major results emerge. First, upon establishment of a temporally constant longitudinal concentration gradient, the cross-sectional concentration distribution is shaped mainly through a balance between longitudinal advective transport and lateral turbulent diffusive mixing. In this context, the longitudinal concentration gradient is assumed to be independent of lateral position. The second result is a relatively simple means for computing the dispersive transport rate, i.e., the mass flux relative to the mean flow. This is obtained by integrating, over the cross section, the product of velocity and concentration, both relative to their cross-sectional means.

For the case of dispersion in fully developed pipe flow, where the velocity distribution is known, Taylor was able to obtain a closed-form result. The corresponding analysis of shear dispersion in a bounded channel is hindered by the lack of a theory for the velocity distribution in the cross section; therefore, some further assumptions must be employed in order to proceed. Observing that a typical cross section of a stream is relatively shallow and wide (typically, depth/width $\cong$ 1:20, at least), that the ratio of vertical to transverse turbulent diffusion coefficient is of order 1:10, and that mixing time scales with ratio (characteristic length)^2/(mixing coefficient), Fischer et al. (1979) concluded that mixing over the depth occurs much faster than mixing across the stream, and consequently, mainly lateral variations of the velocity field need be considered. They proposed as a practical procedure use of the lateral distribution of the depth-averaged values of the vertical velocity profiles.

THEORY

For a prismatic channel of constant depth, the dispersive transport rate turns out to be proportional to the longitudinal concentration gradient, inversely proportional to the depth-averaged transverse turbulent diffusion coefficient, $\langle\varepsilon_T\rangle$, and a function of the integral of the depth-averaged velocity defect, $\langle u\rangle - U$ (local depth-averaged velocity minus its cross-sectional mean). The formulation of the dispersive flux is analogous to Fick's law, with the longitudinal dispersion coefficient, D, as mass transfer coefficient. For a prismatic channel with cross-sectional geometry of variable depth, Fischer et al. (1979) generalize the form of D as follows:

$$D = -\frac{1}{A}\int_0^B \left\{(\langle u\rangle - U)y(\eta)\int_0^\eta \left[\frac{1}{\langle\varepsilon_T\rangle y(\eta)}\int_0^\eta \cdot (\langle u\rangle - U)y(\eta)\,d\eta\right]d\eta\right\}d\eta \quad (2)$$

in which B = the total width; y = the local depth; and η = the transverse variable of integration.

Given velocity measurements and $\langle\varepsilon_T\rangle$, the longitudinal dispersion coefficient is estimable from (2); however, such data are not readily available. To overcome this difficulty, Fischer (1975) developed an important formula for estimating the value of the longitudinal dispersion coefficient from the readily measurable hydraulic quantities depth, width, mean velocity,

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and surface slope, which he structured after the theoretical result of D for a constant depth channel

$$D = I\,\overline{(\langle u\rangle - U)^2}\,\ell^2/\langle \varepsilon_T \rangle \qquad (3)$$

In (3), I is a nondimensional integral, the value of which depends on the velocity distribution; the overbar indicates cross-sectional average; and ℓ = a characteristic length. Fischer selected the value of I as 0.07 and estimated the cross-sectional average of the square of the velocity defect from laboratory measurements to be $0.2U^2$; in consideration of the slightly asymmetric form of typical stream cross sections, he chose $\ell = 0.7B$, and for the mean transverse turbulent diffusion coefficient, he used the value of $\langle \varepsilon_T \rangle = 0.6hu_*$, where h = the average depth; and $u_* = \sqrt{\tau_0/\rho}$ the mean shear velocity on the wetted perimeter. Thus (3) becomes

$$D = \frac{0.011U^2B^2}{hu_*} \qquad (4)$$

Fischer et al. (1979) state that the longitudinal dispersion coefficient can be estimated by (4) within a factor of about four and that, however rough, such estimates are useful in practical work, as results are relatively insensitive to the exact value of D (maximum concentration and plume length depend on $\sqrt{D}$).

The estimator (4) often works well; however, its makeup is not entirely satisfactory. Taylor's result for dispersion in uniform pipe flow gives $D \propto u_*$; yet, in (4), U^2/u_* reduces to $(8/f)u_*$, and the friction factor f is variable. In an effort to improve the foundation of the longitudinal dispersion coefficient estimator, an approach we consider superior to ad hoc empirical ones such as McQuivey and Keefer's (1974), we propose that

$$D = \langle u' \rangle^2 T \qquad (5)$$

where $\langle u' \rangle$ = the average velocity deviation from the cross-sectional mean; and T = the time for cross-sectional mixing. Relationship (5) is indeed equivalent to (3), for $T = \ell^2/\langle \varepsilon_T \rangle$; with Fischer's (1975) assumption, $T = (0.7B)^2/\langle \varepsilon_T \rangle$, and using again $\langle \varepsilon_T \rangle = 0.6hu_*$, (5) gives

$$D = \langle u' \rangle^2 \frac{(0.7B)^2}{0.6hu_*} \qquad (6)$$

The cardinal issue remains the assessment of $\langle u' \rangle$. To resolve this issue, we postulate that the structure of the velocity profile follows von Karman's defect law also in bounded channels

$$u' = u - U = \frac{u_*}{\kappa} F(\zeta) \qquad (7)$$

where ζ = the dimensionless distance from the solid boundary; and κ = the von Karman constant (κ = 0.41 for clear water). This is a working hypothesis whose acceptance leads to the theoretically consistent result that the cross-sectional average of the velocity deviation is proportional to the mean shear velocity. Velocity distributions in natural streams are too complex, however, defying description by a simple analytic function $F(\zeta)$. We therefore write $\langle u' \rangle^2 = \phi(u_*/\kappa)^2$ and use data on dispersion in streams to determine the factor ϕ, capturing in it information on the velocity distribution. This is analogous to estimating in (3) the values of I and of $(\langle u \rangle - U)^2$. The formula for the dispersion coefficient (6) then becomes

$$D = \phi(u^*/\kappa)^2 \frac{(0.7B)^2}{0.6hu^*} \qquad (8)$$

Lumping the constants with ϕ into a single parameter Φ leaves for optimization the formula

$$D = \Phi \frac{u_*B^2}{h} = \Phi \left(\frac{B}{h}\right)^2 u_*h \qquad (9)$$

Christensen (1977) arrived at this form of D in correcting Liu's (1977) empirical formula by dimensional analysis. This relationship was given here a theoretical basis. It can be restated in terms of hydromorphological quantities by relating the mean shear velocity to the hydraulic variables through a macroscopic momentum balance in the flow direction, which yields

$$u_* = \sqrt{gRS} \qquad (10)$$

where g = the gravitational acceleration constant; R = the hydraulic radius (flow area/wetted perimeter); and S = the friction slope ($S \approx \partial h/\partial x \approx$ bed slope). Substituting (10) into (9), we get

$$D = \Phi \frac{\sqrt{gRS}B^2}{h} \qquad (11)$$

ANALYSIS OF DATA

We have optimized Φ using the field data from 16 streams of widely varying characteristics in Table 5.3 of Fischer et al. (1979), which we present in Tables 1 and 2. The mean Φ is 0.6 [compared to Christensen's (1977) value of 0.41, devel-

TABLE 1. Comparison of Measured and Predicted Longitudinal Dispersion Coefficients: Data in British or U.S. Customary Units

River (1)	Depth h (ft) (2)	Shear velocity u_* (ft/s) (3)	Width B (ft) (4)	Mean velocity U (ft/s) (5)	D_{obs} (ft²/s) (6)	Factor in equation (9) $\Phi = Dh/u_*B^2$ (7)	D equation (9) (ft²/s) (8)	Ratio D_{obs}/D (9) (9)	D equation (4) (ft²/s) (10)	Ratio D_{obs}/D (4) (11)
Missouri	10.8	0.26	600	5.1	16,000	1.846	5,200	3.08	36,681	0.44
Clinch	2.8	0.23	155	0.8	150	0.076	1,184	0.13	263	0.57
	7	0.37	195	2.2	600	0.299	1,206	0.50	782	0.77
	6.9	0.36	175	1.5	500	0.313	959	0.52	305	1.64
Bayou Anacoco	3.1	0.22	85	1.1	350	0.683	308	1.14	141	2.48
	3	0.22	120	1.3	425	0.402	634	0.67	406	1.05
Wind/Bighorn	3.6	0.39	195	2.9	450	0.109	2,472	0.18	2,505	0.18
	7.1	0.55	225	5.1	1,750	0.446	2,353	0.74	3,709	0.47
John Day	1.9	0.46	82	3.3	150	0.092	977	0.15	922	0.16
	8.1	0.59	112	2.7	700	0.766	548	1.28	210	3.33
Comite	1.4	0.18	52	1.2	150	0.431	209	0.72	170	0.88
Sabine	6.7	0.18	340	1.9	3,400	1.095	1,863	1.82	3,806	0.89
	15.6	0.27	418	2.1	7,200	2.381	1,814	3.97	2,012	3.58
Yadkin	7.7	0.33	230	1.4	1,200	0.529	1,360	0.88	449	2.67
	12.6	0.42	235	2.5	2,800	1.521	1,105	2.54	717	3.90
Nooksack	9.25	1.7	282	4.1	19,360	1.325	8,769	2.21	935	20.70
	2.5	0.88	210	2.2	375	0.024	9,314	0.04	1,067	0.35

TABLE 2. **Comparison of Measured and Predicted Longitudinal Dispersion Coefficients: Data in SI Units**

River (1)	Depth h (m) (2)	Shear velocity u_* (m/s) (3)	Width B (m) (4)	Mean velocity U (m/s) (5)	D_{obs} (m²/s) (6)	Factor in equation (9) $\Phi = Dh/u_*B^2$ (7)	D equation (9) (m²/s) (8)	Ratio D_{obs}/D (9) (9)	D equation (4) (m²/s) (10)	Ratio D_{obs}/D (4) (11)
Green-Duwamish	1.1	0.049	19.4		9.3	0.555	10.1	0.92	7.7	1.21
Chicago Ship Canal	8.07	0.0191	48.8	0.27	3	0.532	3.4	0.89	12	0.24
Copper Creek (below gauge)	0.49	0.08	16	0.27	20	0.479	25.1	0.80	5.2	3.82
	0.85	0.1	18	0.6	21	0.551	22.9	0.92	15	1.39
	0.49	0.08	16	0.26	9.5	0.227	25.1	0.38	4.9	1.96
Copper Creek (above gauge)	0.4	0.116	19	0.16	9.9	0.095	62.8	0.16	2.2	4.52
Powell	0.85	0.055	34	0.15	9.5	0.127	44.9	0.21	6.1	1.55
Cinch	0.58	0.049	36	0.21	8.1	0.074	65.7	0.12	22	0.37
Coachella Canal	1.56	0.043	24	0.71	9.6	0.605	9.5	1.01	48	0.20

oped from a smaller data base], but values range from 0.074 to 2.4 and give a large standard deviation of about 0.6. We have compiled in Tables 1 and 2 the values of the longitudinal dispersion coefficient predicted by (4) and by (9) with $\Phi = 0.6$, along with the hydraulic quantities on which the predictions are based. The data are presented separately for British and SI units. This was done in order to conform with the data subset presented in the original contribution of Fischer (1975) and to avoid the following unit conversion problem. After consulting that source, we observed that the correspondence between feet and meters in the data of Table 5.3 Fischer et al. (1979) is only an approximate one. The consequence was that application of Fischer's (1975) formula (4) to the data gave results not always in agreement with those presented as predicted by Fischer's formula in Table 5.3 (last column), on occasion varying markedly. For the cases for which no predicted D was given in Table 5.3, we applied (4) with the data in SI units.

For the Green-Duwamish River, the observed D and its estimate by Fischer's formula are taken from the detailed description on page 142 in Fischer et al. (1979). For the Nooksack River, McQuivey and Keefer (1974) present two cases, in which the flow rate varies by one order of magnitude, that yield the unusually high friction factor value $f \approx 1.35$. This figure corresponds to a relative roughness that is abnormally high and outside the range of validity of the standard hydraulic resistance formulas. We are therefore skeptical of the calculated D values shown in the last two rows of Table 1 and have excluded these cases from the calibration of Φ.

Comparison of the results in Tables 1 and 2 shows that: (1) the new formula provides close estimates (within about 25%) in nine cases, as opposed to five by the established formula of Fischer; (2) there are 14 predictions by (9) and 11 by (4) that fall within the range of $0.5 \leq D/D_{pred} \leq 2$ (for a measured D of 100, predicted values ranging from 50 to 200, in any system of units); and (3) there are 7 poor predictions each by (9) (mostly overestimates by factors of about 5–8) and by (4) (under- and overestimates by factors of about 4–6). The average ratio D_{obs}/D_{pred} is 1.0 for (9) and 1.6 for Fischer's formula (4); the corresponding standard deviations are 1.0 and 1.4. These results indicate that the new predictor affords an improvement over the established formula of Fischer. The results shown in Tables 1 and 2, the latter after conversion to British units, are also graphed in Fig. 1.

It is also true, however, that (9) strongly underestimates laboratory flume data, contrary to Fischer's formula, which is based on such data. Nevertheless, this should not be interpreted as failure of (9) to predict dispersion in regular channels adequately, for it performs very well in the cases of the Coachella Canal and Chicago Ship Canal. For the flows in the laboratory flume studied by Fischer, the ratio $(U/u_*)^2 = 8/f$ remains fairly constant: thus, (4) would have the same form as (9), and good agreement would be achieved with $\Phi \neq 0.6$. With $\Phi = 0.6$, estimators (4) and (9) coincide when $U/u_* \approx$

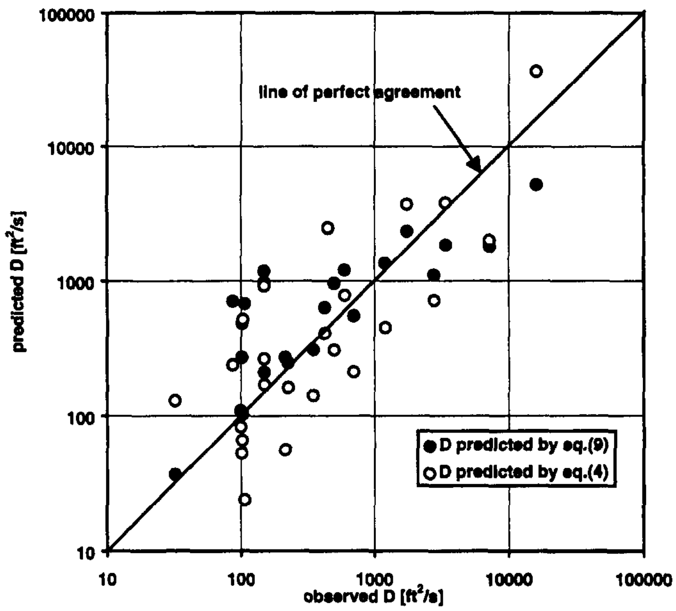

FIG. 1. Comparison of Observed and Predicted Longitudinal Dispersion Coefficients

7.5, or $f \approx 0.03$, which are typical values for open channel flow. For the widely variable flows in natural streams, however, f may range significantly. It is thus unlikely that the values $I = 0.07$ and $\langle u' \rangle^2 = 0.2U^2$, which have been calibrated on flow conditions in laboratory flumes and imply f = const., would also hold, generally, for irregular streams.

Finally, it is useful to check the implications of misestimating D in an engineering application context, e.g., the assessment of pollution from the spill of a conservative contaminant mass M in a stream. The length of the contaminant cloud is $L \approx 4\sqrt{2Dt}$, and the maximum concentration at its center is $C_{max} = M/\sqrt{4\pi Dt}$. In the range $0.5 \leq D/D_{pred} \leq 2$, the relative errors, $\Delta L/L = \sqrt{D_{pred}/D} - 1$ and $\Delta C_{max}/C_{max} = \sqrt{D/D_{pred}} - 1$, are approximately $-0.3 \leq \Delta L/L \leq +0.4$ and $-0.3 \leq \Delta C_{max}/C_{max} \leq +0.4$. These are not insignificant uncertainties; even so, including dispersion in the calculations enhances accuracy in water quality predictions over results of advection models.

CRITIQUE AND LIMITATIONS OF THEORETICAL ESTIMATORS

Theoretical formulas for estimating the dispersion coefficient, such as Fischer's relationship (4) and the relationship (9) developed in this work, are, of course, subject to limitations, for it must be recognized that the great variability of conditions in natural streams does not allow one to capture accurately the details of a complex flow field in a constant within the framework of predictors that use bulk hydraulic quantities. Two of Copper Creek cases (below gauge; see Table 2) demonstrate this limitation; the hydraulic variables are almost identical, yet the measured dispersion coefficients differ by a factor of two. Although an experienced river engineer could examine the site-specific hydromechanic conditions and improve transport predictions, it is always advisable to use estimates of pollutant concentrations based on dispersion coefficient equations such as (4) or (9) with caution.

Differences between measured and predicted dispersion coefficients are also due to temporary storage in cavities, groins, and dead storage zones in the course of the streams. The effects of these features are not included in (4), as laid out clearly by Fischer et al. (1979), and are only accounted implicitly and incompletely in (9) via the field calibration of Φ. Furthermore, the predictors (4) and (9) ignore the effect of stream meanders, for which Fischer et al. (1979) proposed an approximation. Finally, as mentioned at the outset, the one-dimensional approach, on which the longitudinal dispersion coefficient concept is based, is valid only after an initial period, during which the transport is two-dimensional. Knowing the extent of the initial region is important in the design of field studies; the measuring stations should be placed in the dispersive region where (1) applies. Fischer et al. (1979) suggest that the one-dimensional approximation is adequate for $x > 0.4UB^2/\langle\varepsilon_T\rangle$, a distance that can be substantial; however, Holley and Jirka (1986, Chapters 6 and 7) indicate that this limit may be conservative.

SUMMARY AND CONCLUSIONS

We have presented a predictor for the longitudinal dispersion coefficient for streams that follows Fischer's original contribution and is based on readily measurable hydraulic quantities. The predictor is structured after the fundamental formula of G. I. Taylor for dispersion in fully developed pipe flow and entails a free parameter that intends to capture information on the cross-sectional velocity distribution through correlation with data from dispersion studies in streams. Comparison of predicted with observed values shows the new optimized formula providing closer estimates than the established formula of Fischer with respect to mean accuracy and to deviations from the mean, achieving in 14 of the 24 cases studied predictive accuracy of $0.5 \le D_{obs}/D_{pred} \le 2$. Yet despite the progress made, reliable prediction of the value of the longitudinal dispersion coefficient for streams remains an open research topic.

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APPENDIX II. NOTATION

The following symbols are used in this paper:

A = cross-sectional area of stream;
B = surface width of stream;
C = cross-sectionally averaged solute concentration;
D = longitudinal dispersion coefficient;
F = function;
f = friction factor;
g = constant of gravitational acceleration;
h = mean cross-sectional depth;
I = Fischer's nondimensional integral;
ℓ = characteristic length;
L = length of contaminant cloud;
R = hydraulic radius;
T = time for cross-sectional mixing;
t = time;
U = mean cross-sectional velocity;
u = velocity at a point in the cross section;
u_* = shear velocity;
u' = local velocity deviation from the cross-sectional mean value;
x = distance along the thalweg;
y = local depth;
ε_T = transverse turbulent diffusion coefficient;
ζ = distance from solid boundary;
η = variable of integration in the transverse direction;
θ = angle of inclination of the channel bed against the horizontal;
κ = von Karman's constant;
ρ = mass density of water;
τ_0 = wall shear stress;
ϕ = numerical parameter in estimator of the longitudinal dispersion coefficient; and
Φ = numerical parameter in final formula for the longitudinal dispersion coefficient.