

The pollutant settles ($v = 0.11 \text{ m d}^{-1}$). What mass loading could be input to this system under steady-state conditions if the allowable concentration at the outfall is 10 ppb? Express your results in kg d^{-1} . Assume complete lateral and vertical mixing at the outfall.

9.6. A point source is discharged into an estuary having the following characteristics:

	Value	Units
Dispersion coefficient	10^6	$\text{m}^2 \text{d}^{-1}$
Flow	5×10^4	$\text{m}^3 \text{d}^{-1}$
Width	200	m
Depth	2	m

The pollutant decays at a rate of 0.2 d^{-1} at $T = 20^\circ\text{C}$ with $Q_{10} = 1.7$. The estuary has an ambient temperature of 27.5°C .

- (a) Determine the reaction rate for the ambient temperature.
 (b) What mass loading could be input to this system under steady-state conditions if the allowable concentration at the outfall is 20 ppb? Express your results in kg d^{-1} . Assume complete lateral and vertical mixing at the outfall.

9.7. A stream receives a point and a diffuse source as shown in Fig. P9.7:

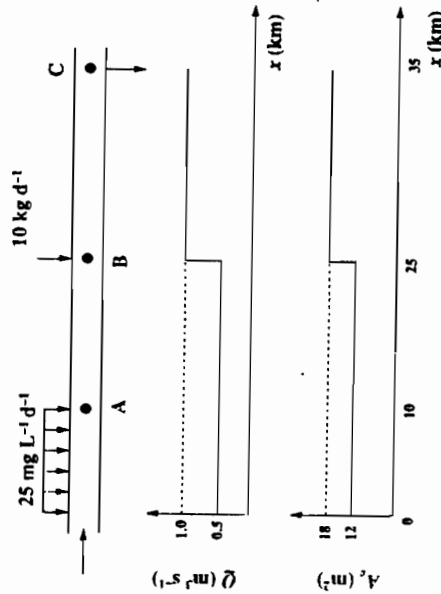


FIGURE P9.7

The boundary condition (at $x = 0$) is $c = 10 \text{ mg L}^{-1}$. If the pollutant decays at a rate of 0.2 d^{-1} , determine the steady-state concentrations at points A, B, and C.

9.8. Rederive Eqs. 9.56 and 9.57 for the case where the loading contributes a significant flow to the estuary.

9.9. Rederive Eq. 9.57 so that it converges to the plug-flow solution at low dispersion. Note that one way to do this utilizes the following alternative formula for determining the roots of a quadratic (Chapra and Canale 1988):

$$x = \frac{-c}{b \pm \sqrt{b^2 - 4ac}} \quad (9.61)$$

Distributed Systems (Time-Variable)

LECTURE OBJECTIVE: I continue our discussion of distributed systems by studying the temporal characteristics of plug-flow and mixed-flow systems. We focus on the instantaneous discharge into a one-dimensional channel. Such models are useful for modeling spills and for tracer studies in streams and estuaries.

I now describe some models for simulating time-variable changes in distributed systems. As in the previous lecture, the discussion is limited to one-dimensional systems with constant characteristics and to the instantaneous discharge of a substance into such channels.

The resulting models are extremely useful in environmental engineering. In particular they can be used to simulate an accidental spill in a stream or an estuary. In addition they can be used to analyze tracer studies, that is, for those cases where we deliberately introduce a tracer (such as a dye) into a water body to estimate some of its characteristics (for example velocity, dispersion, reaction rate).

10.1 PLUG FLOW

The time-variable mass balance for the plug-flow system can be written as (Fig. 10.1)

$$\frac{\partial c}{\partial t} = -U \frac{\partial c}{\partial x} - kc \quad (10.1)$$

As derived in Box 10.1, if a spill causes a concentration c_0 at $t = x = 0$, the solution is

$$c = c_0 e^{-kt} \quad \text{for } t = x/U$$

$$c = 0 \quad \text{otherwise} \quad (10.2)$$

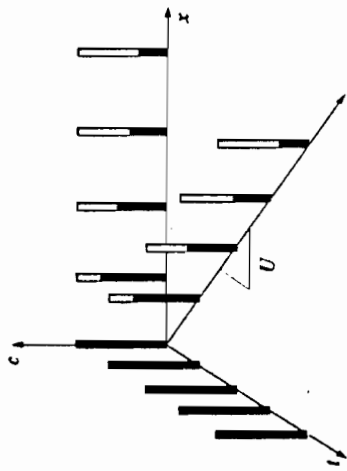


FIGURE 10.1

Depiction of the movement of dye in space and time for a plug-flow system. The whole plugs show the movement of a conservative substance. The shaded portions show a substance that reacts with first-order kinetics.

Further, the velocity establishes a direct relationship between time and space,

$$t = \frac{x}{U} \quad (10.3)$$

That is, the solution can also be written as

$$c = c_0 e^{-k/U x} \quad \text{for } x = Ut \quad (10.4)$$

$$c = 0 \quad \text{otherwise}$$

BOX 10.1. Characteristics of the Plug-Flow Equation

The time-variable mass balance for the plug-flow system can be written as

$$\frac{\partial c}{\partial t} + U \frac{\partial c}{\partial x} = -kc \quad (10.5)$$

Now suppose that we want to find a solution to this equation along an arbitrary curve in the x - t plane (Fig. B10.1). The change of c , dc , from points A to B can be written as

$$dc = \frac{\partial c}{\partial t} dt + \frac{\partial c}{\partial x} dx \quad (10.6)$$

Dividing this equation by dt gives

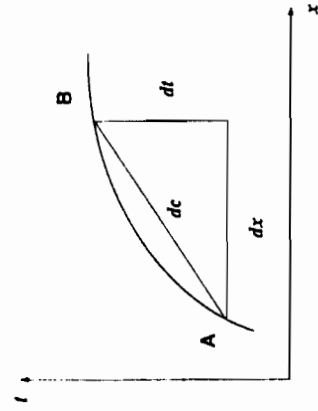


FIGURE B10.1

$$\frac{dc}{dt} = \frac{\partial c}{\partial t} + \frac{\partial c}{\partial x} \frac{dx}{dt} \quad (10.7)$$

where dx/dt is the slope of the curve AB in the x - t plane. The term dc/dt on the left-hand side represents the rate of change as measured by a moving observer. The term $\partial c/\partial t$ represents the change of c at a fixed position. The term $(dx/dt)(\partial c/\partial x)$ represents the change due to the observer's moving into a region of possibly different c .

Now suppose that the curve is chosen so that its slope is equal to U . That is,

$$\frac{dx}{dt} = U \quad (10.8)$$

In other words suppose that the observer is moving at the same rate as the conservative velocity U . If this is true Eq. 10.7 becomes

$$\frac{dc}{dt} = \frac{\partial c}{\partial t} + U \frac{\partial c}{\partial x} \quad (10.9)$$

Thus the right-hand side of Eq. 10.9 is equal to the left-hand side of Eq. 10.5. Consequently

$$\frac{dc}{dt} = -kc \quad (10.10)$$

This is a nice result. In essence we have converted the original partial differential equation into a pair of ordinary differential equations (10.8 and 10.10). The former (10.8) represents a curve in the x - t plane that is called the *characteristic curve*. In the present case it is a straight line that has a slope of U . If we assume that $t = 0$ at $x = 0$ (that is, these are the coordinates where the observer starts to move downstream with the velocity), Eq. 10.8 can be integrated to yield

$$x = Ut \quad (10.11)$$

Thus the velocity defines a linear relationship between space and time. That is, at time t the observer will have moved a distance Ut downstream.

The concentration seen by the observer is then obtained by integrating Eq. 10.10 along this curve. For example if $c = c_0$ at $t = 0$,

$$c = c_0 e^{-kt} \quad (10.12)$$

Consequently, as depicted in Fig. 10.1, the solution to the original partial differential equation is an exponential decay that occurs along the line defined by Eq. 10.11.

EXAMPLE 10.1. SPILL INTO A PLUG-FLOW SYSTEM. Five kg of a conservative pollutant is spilled into a stream over a period of about 5 min. The stream has the following characteristics: flow = $2 \text{ m}^3 \text{ s}^{-1}$ and cross-sectional area = 10 m^2 . Determine the concentration and the extent of the spill and how long it takes to reach a water intake located 6.48 km downstream.

Solution: The spill concentration can be estimated as follows:

$$W_{\text{spill}} = \frac{5 \text{ kg}}{5 \text{ min}} \left(\frac{1000 \text{ g}}{\text{kg}} \frac{1440 \text{ min}}{\text{d}} \right) = 1.44 \times 10^6 \text{ g d}^{-1}$$

$$Q = 2 \text{ m}^3 \text{ s}^{-1} \left(\frac{86,400 \text{ s}}{\text{d}} \right) = 0.1728 \times 10^6 \text{ m}^3 \text{ d}^{-1}$$

$$c = \frac{W_{\text{spill}}}{Q} = \frac{1.44 \times 10^6}{0.1728 \times 10^6} = 8.33 \text{ g m}^{-3}$$

The extent of the spill in this stretch is calculated as (note, $U = 2/10 = 0.2 \text{ m s}^{-1}$)

$$x_{\text{spill}} = 5 \text{ min}(0.2 \text{ m s}^{-1}) \left(\frac{60 \text{ s}}{\text{min}} \right) = 60 \text{ m}$$

The front of the spill will reach the water intake at

$$t = \frac{6480 \text{ m}}{0.2 \text{ m s}^{-1}} \left(\frac{\text{hr}}{3600 \text{ s}} \right) = 9 \text{ hr}$$

from the start of the spill. Five minutes later it will have passed beyond the intake.

The preceding example took the perspective of an observer at a fixed location x downstream from the spill. This observer will see nothing until $t = x/U$. At this instant the spill will pass the observation point. For the case where the spill is decaying, the concentration will be reduced as specified by the exponential decay in Eq. 10.2.

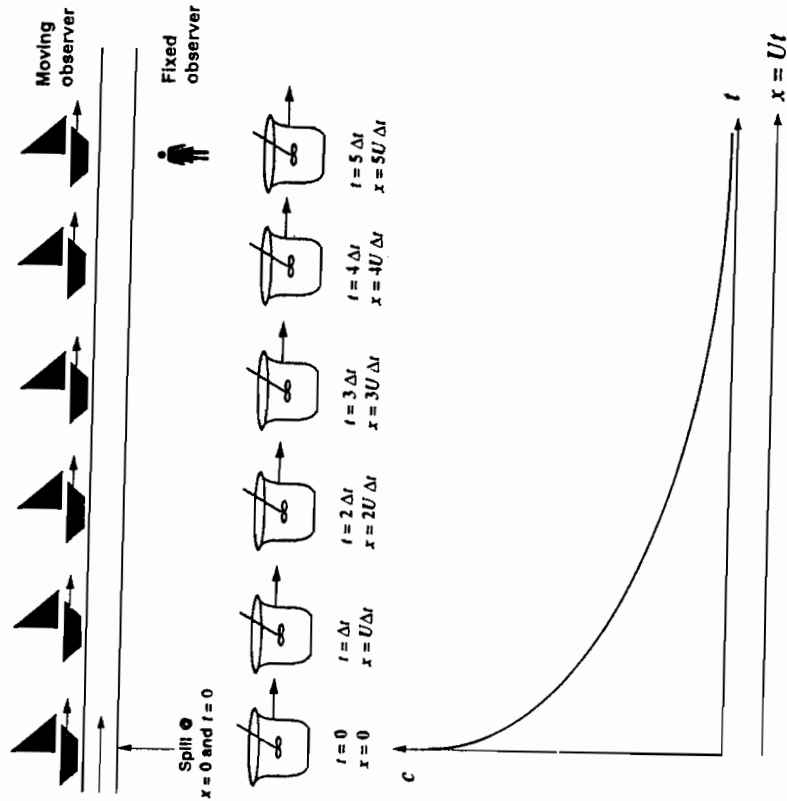


FIGURE 10.2
Two perspectives for viewing temporal and spatial changes in a plug-flow system.

An alternative perspective exists for a moving observer who travels downstream at the same velocity as the spill. As depicted in Fig. 10.2 this observer would watch the concentration decay exponentially as if the process were taking place in a batch reactor.

The interrelationship between space and time also applies to steady-state solutions. For example recall that the steady-state solution for a continuous load into a plug-flow system is

$$c = c_0 e^{-kx/U} \quad (10.13)$$

Now we should recognize that this equation could also be written as

$$c = c_0 e^{-kt} \quad (10.14)$$

where t represents travel time below the point source. The interchangeability of space and time will recur in our subsequent analyses of plug-flow systems throughout the remainder of this book.

Clearly the foregoing discussion is a highly idealized representation of stream transport. In even the most highly advective system, some dispersion will occur. Before showing how the dispersive process can be modeled, I'll first introduce the random walk model of the diffusion/dispersion process.

10.2 RANDOM (OR "DRUNKARD'S") WALK

The *random* or "*drunkard's*" *walk* is a term used to describe the type of random motion found in many diffusion processes. The name "*drunkard's* walk" stems from the similarity between this type of motion and the random stumbling around that a drunkard might exhibit.

Suppose that a population of particles is confined to motion along a one-dimensional line (Fig. 10.3). Assume that each particle has an equal likelihood of moving a small distance Δx to either the left or the right over a time interval Δt . At $t = 0$ all particles are grouped at $x = 0$ and are allowed to take one random step in either direction. After Δt has elapsed, approximately one-half of the particles would have stepped to the right (Δx) and the remainder would have stepped left ($-\Delta x$). After another time interval (that is, after $2\Delta t$ had elapsed) approximately one-fourth would be at $-2\Delta x$, one-fourth at $2\Delta x$, and one-half would have stepped back to the origin.

With additional time the particles would spread out (Fig. 10.3). Note that the distribution of the population is not uniform but has a higher density at the origin and diminishes at the end. This is due to the fact that a particle would have to execute many successive moves in a single direction to reach the extremes. For example in Figure 10.3, after $4\Delta t$, a particle would have to execute 4 successive right steps to reach $4\Delta x$. Because there is a 50-50 chance of moving left or right in each time interval, it is more likely that a particle would stumble around in the vicinity of the origin. The net outcome is that the random walk of the individual particles results in a spreading bell-shaped distribution of the population. In addition note that this spreading tendency amounts to a general movement of particles from high to low concentrations.

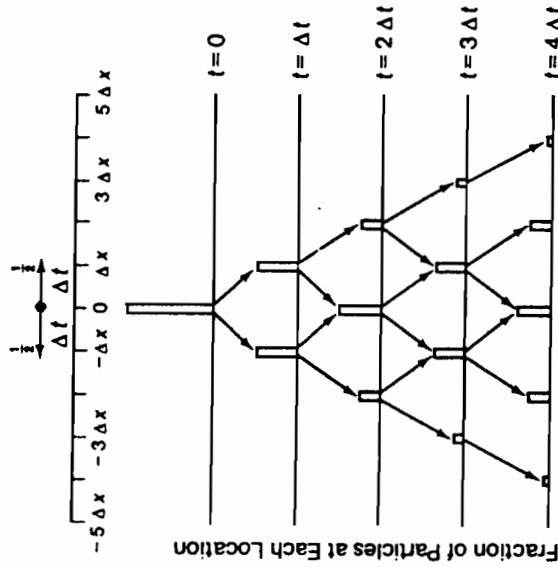


FIGURE 10.3
Graphic representation of a random walk. At time $t = 0$, all particles are grouped at the origin ($x = 0$). During each time step Δt , half the particles at each location move left and half move right. The result is that over time the particles spread out in a bell-shaped pattern.

The random-walk process can be expressed mathematically as (see Box 10.2 for details)

$$p(x, t) = \frac{1}{2\sqrt{\pi Dt}} e^{-\frac{x^2}{4Dt}} \quad (10.15)$$

where $p(x, t)$ = the probability that a particle will be at x after an elapsed time t and D = a diffusion coefficient, defined as

$$D = \frac{\Delta x^2}{2\Delta t} \quad (10.16)$$

If the population is grouped at the origin at time zero, the number of individuals at position x at a subsequent time t would be proportional to the probability of an individual particle's being at x . Thus Eq. 10.15 can be expressed in terms of mass and concentration,

$$c(x, t) = \frac{m_p}{2\sqrt{\pi Dt}} e^{-\frac{x^2}{4Dt}} \quad (10.17)$$

where m_p = total mass of the particles normalized to the cross-sectional area

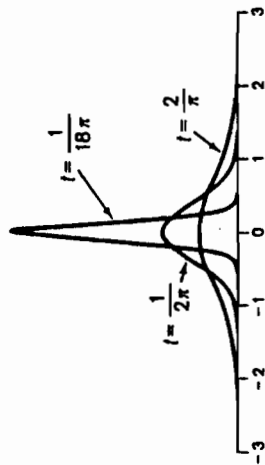


FIGURE 10.4
Representation of the random walk by a normal or "bell-shaped" distribution.

($M L^{-2}$).[†] Thus the distribution of the population of particles is described by a series of bell-shaped curves that spread out over time symmetrically about the origin (Fig. 10.4).

It should be noted that Eqs. 10.15 to 10.17 apply only if the number of time steps and spatial intervals are much greater than 1. In other words the observation time t and the observation space x must be much greater than the magnitudes of the duration Δt and size Δx of the individual steps. Because the random walk is an analog of the diffusion process, these conditions will have relevance to the application of diffusion models in the real world. That is, the process holds only when the random motions, whether molecular or due to eddies, have smaller time and space scales than do the phenomena being modeled.

BOX 10.2. Mathematics of the Random Walk

The random walk can be formulated mathematically by realizing that it can be represented as a binomial distribution. The *binomial* or *Bernoulli's distribution* results from our tendency to place observations in one or the other of two mutually exclusive categories. If p is the probability that an event will occur and $q = 1 - p$ is the probability that it will not, the probability that the event will happen exactly x times in n trials is given by

$$P(x, n) = \binom{n}{n-x} p^x q^{n-x}$$

where the parenthetical operation represents the number of possible ways or combinations by which the event can occur x times in n trials, where

$$\binom{n}{n-x} = \frac{n!}{x!(n-x)!}$$

and $n! = n(n-1)(n-2) \dots 1$.

The random walk is this sort of process because the particle is limited to two modes of motion: left or right. Thus the binomial distribution can be used to determine the probability that, after n_t time steps, the particle will be n_x spatial intervals from the origin. To do this the total number of steps would be divided into n_r to the right and

[†]Note that m_p is formally called a *plane source* since it enters the system at a plane surface—the cross-sectional area.

$n_r - n_l$ to the left. To wind up at n_x , the difference between the right and left steps, $n_r - (n_l - n_r)$, would have to equal n_x . Therefore the number of right steps would have to be $n_r = (n_l + n_x)/2$ and left steps would have to be $(n_l - n_r) = (n_l - n_x)/2$. The probability that after n_l time steps the particle would be n_x spatial intervals from the origin can be represented as (remember there is an equal likelihood of moving left or right; that is, $p = q = 0.5$)

$$p(n_r, n_l) = \binom{n_l}{n_r + n_l} \left(\frac{1}{2}\right)^{\frac{n_r + n_l}{2}} \left(\frac{1}{2}\right)^{\frac{n_l - n_r}{2}}$$

$$\text{or } p(n_r, n_l) = \binom{n_l}{\frac{n_l + n_x}{2}} \left(\frac{1}{2}\right)^{\frac{n_l + n_x}{2}} \left(\frac{1}{2}\right)^{\frac{n_l - n_x}{2}}$$

As the number of intervals becomes very large, the binomial distribution approaches the normal distribution. If we define continuous variables $x = n_x \Delta x$ and $t = n_l \Delta t$, the probability that the particle would be at distance x at time t can be formulated as a normal distribution with a mean of zero and a variance of $t(\Delta x)^2/\Delta t$, as in (see Pielou 1969)

$$p(x, t) = \frac{1}{\sqrt{2\pi Dt}} e^{-\frac{x^2}{2Dt}}$$

where $p(x, t)$ = the probability that a particle will be at x after an elapsed time t and D = a diffusion coefficient, defined in the limit (as Δx and Δt become small) as

$$D = \frac{\Delta x^2}{2 \Delta t}$$

10.3 SPILL MODELS

Now that we have reviewed the fundamentals of the random-walk model, we can integrate dispersion into models of instantaneous discharges into uniform, one-dimensional channels. First, we investigate the case where the spill occurs instantaneously. Then we present a model for a continuous input.

10.3.1 Instantaneous or "Impulse" Spills

For this case a significant input occurs over a very short time period at a point in space. Many spills can be approximated in this fashion. In the following paragraphs we build the solution by adding mechanisms on a term-by-term basis. First, we examine the situation where dispersion or turbulent diffusion is the only mechanism.

Diffusion/dispersion. As developed in Lec. 9, a mass balance for a substance that disperses in a one-dimensional channel can be written as (Eq. 9.21 with $k = U = 0$)

$$\frac{\partial c}{\partial t} = E \frac{\partial^2 c}{\partial x^2} \tag{10.18}$$

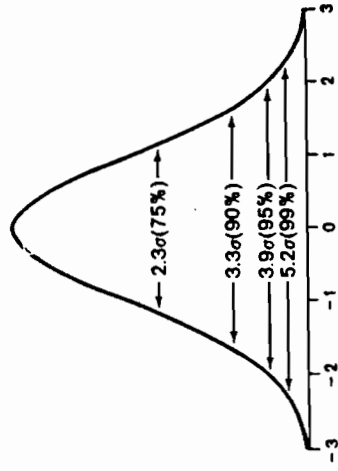


FIGURE 10.5
A standardized normal distribution showing the probability (expressed as percent) encompassed by various multiples of the standard deviation. For example 3.9σ encompasses 95% of the area under the curve.

This relationship is sometimes referred to as *Fick's second law*. The solution for the case where the substance is initially concentrated at $x = 0$ is

$$c(x, t) = \frac{m_p}{2\sqrt{\pi Et}} e^{-\frac{x^2}{4Et}} \tag{10.19}$$

which is identical to the solution based on the random walk (Eq. 10.17). Thus the solution is a bell-shaped curve with mean at zero and a variance of $2Et$.

Because the variance is a measure of spread, this equation can be used as the basis of a simple engineering computation to assess the effect of dispersion on a pollutant spill. For example if a conservative substance were discharged in a lump sum to a water body, its tendency to spread outward from its center of mass could be represented by the standard deviation

$$\sigma = \sqrt{2Et} \tag{10.20}$$

or multiples of the standard deviation as depicted in Fig. 10.5. For example 95% and 99% spreads can be roughly approximated by 4σ and 5σ , respectively.

EXAMPLE 10.2. CONSERVATIVE SPILL IN A CHANNEL WITH NO FLOW.

A barge releases a large quantity of a highly persistent contaminant in the center of a canal that is not flowing. If the dispersion coefficient is approximately $10^5 \text{ m}^2 \text{ d}^{-1}$, how far will the contaminant spread in 1 d? In 2 d? Assume that a 95% band adequately approximates the extent of the spill.

Solution: Using Eq. 10.20 and the multiple of the standard deviation that encompasses 95% of the distribution (Fig. 10.5) yields

$$x(1 \text{ d}) = 3.9 \sqrt{2(10^5)} = 1744 \text{ m}$$

and $x(2 \text{ d}) = 3.9 \sqrt{2(10^5)2} = 2466 \text{ m}$

Dispersion/advection. Now we can add advection to the model. This relationship is sometimes referred to as the *advection-diffusion* (or *advection-dispersion*) equation

$$\frac{\partial c}{\partial t} = -U \frac{\partial c}{\partial x} + E \frac{\partial^2 c}{\partial x^2} \tag{10.21}$$

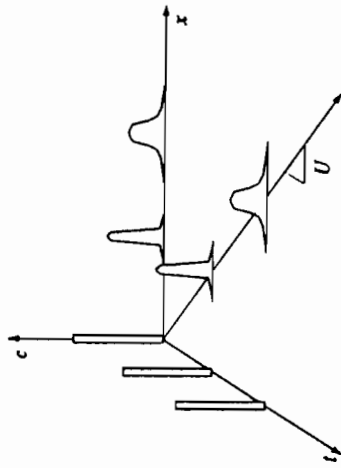


FIGURE 10.6
The movement of conservative dye in space and time for a mixed-flow system.

The solution for the case where the substance is initially concentrated at $x = 0$ is

$$c(x, t) = \frac{m_p}{2\sqrt{\pi Et}} e^{-\frac{(x-Ut)^2}{4Et}} \quad (10.22)$$

Note that in comparison with Eq. 10.19, the effect of advection is to “move” the dispersion solution intact downstream at velocity U (Fig. 10.6).

EXAMPLE 10.3. SPILL INTO A MIXED-FLOW SYSTEM. Evaluate the spill from Example 10.1, but include the effect of dispersion. Assume that a dispersion coefficient of $0.1 \text{ m}^2 \text{ s}^{-1}$ holds for the entire stretch. Also assume that the spill occurs instantaneously as a plane source.

Solution: The spill can be expressed as a plane source, as in

$$m_p = \frac{5 \times 10^3 \text{ g}}{10 \text{ m}^2} = 500 \text{ g m}^{-2}$$

The spill concentration in the stretch can then be estimated by

$$c(x, t) = \frac{500}{2\sqrt{\pi(0.1)t}} e^{-\frac{(x-0.1t)^2}{4(0.1)t}}$$

Results can be calculated and are displayed in Fig. E10.3. These results are very different from Example 10.1. Although the travel time to the water intake is the

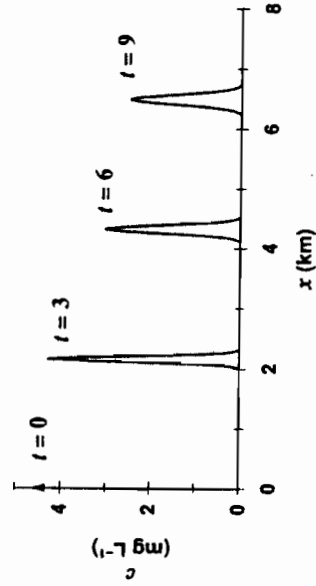


FIGURE E10.3

same, the peak progressively decays due to dispersion. In addition the spill spreads as time progresses. These results can be summarized as

	$t = 3 \text{ hr}$	$t = 6 \text{ hr}$	$t = 9 \text{ hr}$
Extent of spill (m) ^a	181	256	314
Peak concentration (mg L ⁻¹)	4.29	3.03	2.48

^a Defined as distance encompassing 95% of the mass.

Dispersion/advection/decay. Finally a first-order reaction can be added to the model,

$$\frac{\partial c}{\partial t} = -U \frac{\partial c}{\partial x} + E \frac{\partial^2 c}{\partial x^2} - kc \quad (10.23)$$

The solution for the case where the substance is initially concentrated at $x = 0$ is

$$c(x, t) = \frac{m_p}{2\sqrt{\pi Et}} e^{-\frac{(x-Ut)^2}{4Et} - kt} \quad (10.24)$$

In comparison with Eq. 10.22, the effect of decay reduces to the area under the bell-shaped curve as it moves downstream (Fig. 10.7).

Fixed versus global observer. Note that Eq. 10.24 is a function of two independent variables x and t . Thus it can be viewed from two perspectives. As in Example 10.3 we can compute the spatial distribution at a fixed time. Thus we can develop the global perspective manifested by the bell-shaped curves shown in Fig. 10.7.

Conversely we can compute the temporal distribution at a fixed point in space. This latter case relates to a static observer. For this case the view can be skewed because the bell-shaped curve continues to spread out as it is being observed (Fig. 10.8).

10.3.2 Continuous Spills

For some tracer studies, as well as some spills, the input jumps to a constant level. As depicted in Fig. 10.9, two idealized cases can be modeled. First, the concentration

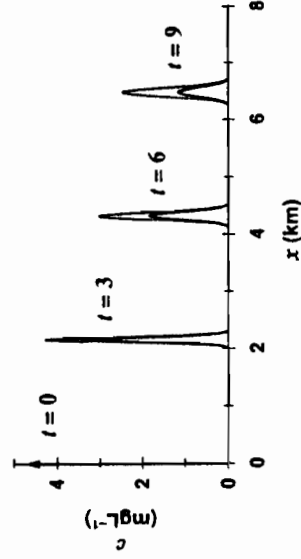


FIGURE 10.7
The effect of decay on the spill model. The thin lines are the same as calculated in Example 10.3. The heavy lines are for the same case but with $k = 2 \text{ d}^{-1}$.

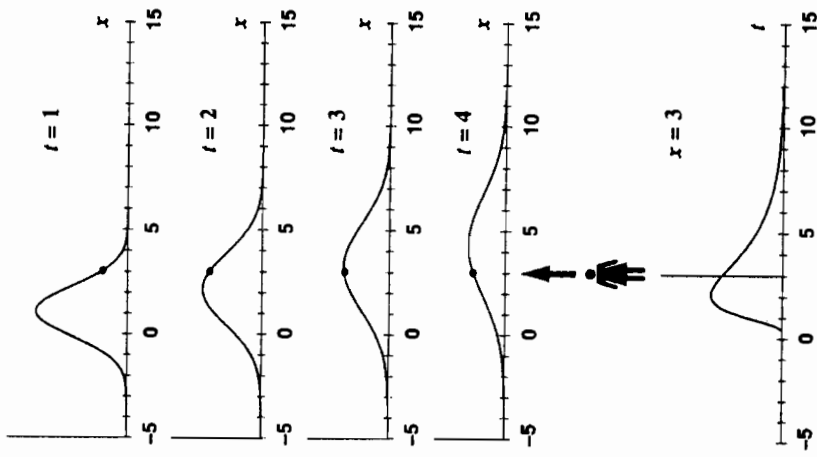


FIGURE 10.8 Although the distribution of a pollutant spill is bell-shaped in space, a fixed observer would "see" a skewed shape in time because the bell-shaped curve continues to spread out as it is being observed.

increase can be maintained for an infinite duration (Fig. 10.9a). For this case the solution of Eq. 10.23 with constant coefficients can be expressed as (O'Loughlin and Bowmer 1975)

$$c(x, t) = \frac{c_0}{2} \left[\frac{Ux(1-\Gamma)}{e^{2Et}} \operatorname{erfc} \left(\frac{x - Ut\Gamma}{2\sqrt{Et}} \right) + \frac{Ux(1+\Gamma)}{e^{2Et}} \operatorname{erfc} \left(\frac{x + Ut\Gamma}{2\sqrt{Et}} \right) \right] \quad (10.25)$$

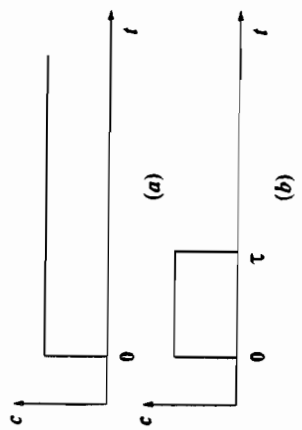


FIGURE 10.9 Continuous inputs are characterized in two ways: (a) infinite and (b) finite durations.

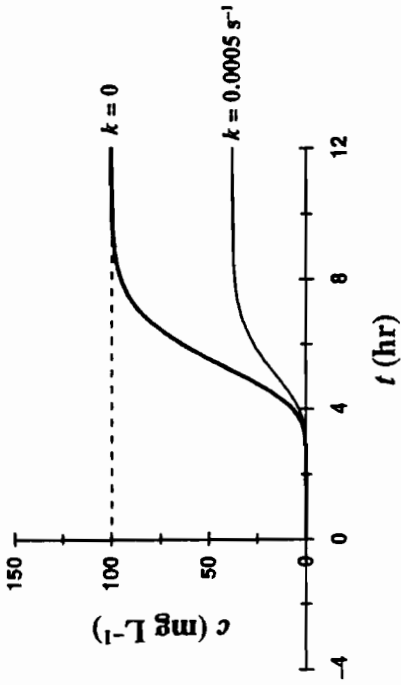


FIGURE 10.10 A simulation of a "breakthrough" curve. A step increase to a concentration of 100 mg L⁻¹ is initiated at x = 0 at t = 0. Shown is the change in concentration at a sampling point 2000 m downstream for a conservative and a nonconservative release. This example used U = 0.1 m s⁻¹ and E = 5 m² s⁻¹.

where

$$\Gamma = \sqrt{1 + 4\eta} \quad (10.26)$$

and
$$\eta = \frac{kE}{U^2} \quad (10.27)$$

The error function complement, erfc, is equal to one minus the error function: 1 - erf. Also, erf(-x) = -erf(x). The error function is simply the evaluation of the following definite integral,

$$\operatorname{erf}(b) = \frac{2}{\sqrt{\pi}} \int_0^b e^{-\beta^2} d\beta \quad (10.28)$$

where β = a dummy variable. We have included selected values for the error function in App. G. It should also be noted that the error function is available on standard software libraries [e.g., the International Math and Statistics Library (IMSL), Numerical Recipes (Press et al. 1992, etc.), and as a function on many software packages (e.g., Excel, Mathematica, etc.)].

A simulation using Eq. 10.25 is shown in Fig. 10.10. Such curves are referred to as "breakthrough" curves. They are used extensively in both surface and groundwater problem contexts.

The second idealized application applies to cases in which the step input terminates after a finite time (Fig. 10.9b). For this situation the solution is presented in two parts. For t < τ, Eq. 10.25 holds. Thereafter the following formula applies (Runkel 1996):

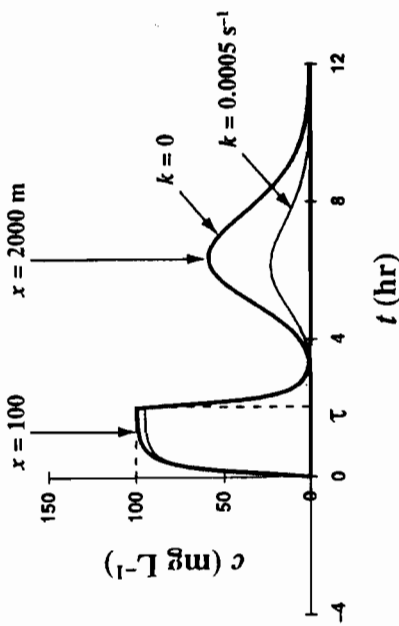


FIGURE 10.11
A simulation of a dye release or spill of finite duration—a “breakthrough” curve. A step increase to a concentration of 100 mg L⁻¹ is initiated at x = 0 at t = 0 and lasts for τ = 2 hr. Shown is the distribution at x = 0 (dashed line) along with curves at x = 100 and 2000 m for a conservative (bold line) and a nonconservative release (light line). This example used U = 0.1 m s⁻¹ and E = 5 m² s⁻¹.

$$c(x, t) = \frac{c_0}{2} \left\{ e^{\frac{Ux}{2E}(1-\Gamma)} \left[\operatorname{erfc} \left(\frac{x - Ut}{2\sqrt{Et}} \right) - \operatorname{erfc} \left(\frac{x - U(t - \tau)}{2\sqrt{E(t - \tau)}} \right) \right] + e^{\frac{Ux}{2E}(1+\Gamma)} \left[\operatorname{erfc} \left(\frac{x + Ut}{2\sqrt{Et}} \right) - \operatorname{erfc} \left(\frac{x + U(t - \tau)}{2\sqrt{E(t - \tau)}} \right) \right] \right\} \quad (10.29)$$

A simulation using Eq. 10.25 is shown in Fig. 10.11. Notice how the solution approaches a bell-shaped curve as it moves downstream.

10.4 TRACER STUDIES

Aside from accidental spills, the models determined above have utility when compounds are deliberately discharged as in a tracer study. In such cases the distribution downstream from the injection point can be used to determine key characteristics such as the velocity, the dispersion coefficient, and the decay rate.

To do this it is necessary to estimate some quantities from concentration data. Such data is usually measured at discrete points in time (Fig. 10.12).

For such cases the following formulas can be used:

Mean Concentration

$$\bar{c} = \frac{\sum_{i=0}^{n-1} (c_i + c_{i+1})(t_{i+1} - t_i)}{2(t_n - t_0)} \quad (10.30)$$

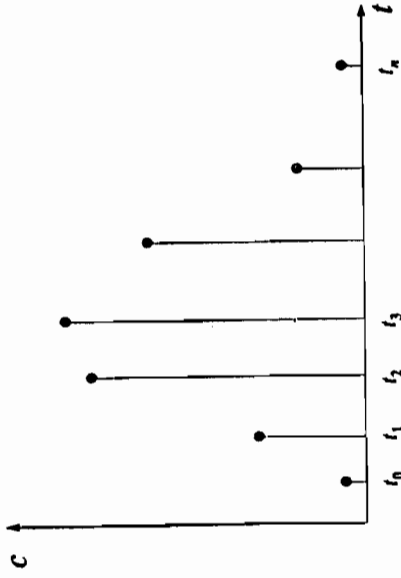


FIGURE 10.12
Concentration data sampled at a point in space to characterize the distribution of a tracer.

Mass $M = Q\bar{c}(t_n - t_0) \quad (10.31)$

Travel Time
$$\bar{t} = \frac{\sum_{i=0}^{n-1} (c_i t_i + c_{i+1} t_{i+1})(t_{i+1} - t_i)}{\sum_{i=0}^{n-1} (c_i + c_{i+1})(t_{i+1} - t_i)} \quad (10.32)$$

Temporal Variance
$$s_t^2 = \frac{\sum_{i=0}^{n-1} (c_i t_i^2 + c_{i+1} t_{i+1}^2)(t_{i+1} - t_i)}{\sum_{i=0}^{n-1} (c_i + c_{i+1})(t_{i+1} - t_i)} - (\bar{t})^2 \quad (10.33)$$

If data are available from two stations, located at x_1 and x_2 , the mean velocity can be estimated by

$$U = \frac{x_2 - x_1}{t_2 - t_1} \quad (10.34)$$

The velocity estimate can, in turn, be used to calculate the dispersion coefficient by (Fischer 1968)

$$E = \frac{U^2 (s_t^2 - s_x^2)}{2(t_2 - t_1)} \quad (10.35)$$

EXAMPLE 10.4. EVALUATION OF A TRACER STUDY. A tracer study is conducted in a stream with a flow of $3 \times 10^4 \text{ m}^3 \text{ d}^{-1}$ and a width of 45 m. At $t = 0$, 5 kg of

a conservative substance, lithium, is instantaneously injected at $x = 0$. Concentrations are measured at two downstream stations:

$$x = 1 \text{ km}$$

t (min)	30	40	50	60	70	80	90	100	110	120
Lithium ($\mu\text{g L}^{-1}$)	0	100	580	840	560	230	70	15	3	0

$$x = 8 \text{ km}$$

t (min)	370	400	430	460	490	520	550	580	610
Lithium ($\mu\text{g L}^{-1}$)	0	10	80	250	280	140	35	5	0

Determine (a) the velocity (m d^{-1}) and (b) the dispersion coefficient ($\text{cm}^2 \text{s}^{-1}$).

Solution: (a) The velocity can be evaluated by determining the travel time between the two sampling points,

$$\bar{t}_1 = \frac{2,982,600}{47,960} = 62.2 \text{ min}$$

$$\bar{t}_8 = \frac{23,133,000}{48,000} = 481.9 \text{ min}$$

Therefore the velocity can be computed as

$$U = \frac{8 - 1 \text{ km}}{481.9 - 62.2 \text{ min}} \left(\frac{1000 \text{ m}}{\text{km}} \right) = 16.67 \text{ m min}^{-1} \left(\frac{1440 \text{ min}}{\text{d}} \right) = 24,014 \text{ m d}^{-1}$$

(b) The dispersion coefficient can be estimated by calculating the time variances of the lithium concentration distributions,

$$s_{t1}^2 = \frac{1.92 \times 10^8}{47,960} - 62.2^2 = 137 \text{ min}^2$$

$$s_{t2}^2 = \frac{1.12 \times 10^{10}}{48,000} - 481.9^2 = 1043 \text{ min}^2$$

Then Eq. 10.35 can be used to compute

$$E = \frac{(16.67 \text{ m min}^{-1})^2 (1043 \text{ min}^2 - 137 \text{ min}^2)}{2(481.9 \text{ min} - 62.2 \text{ min})} \\ = 300 \text{ m}^2 \text{ min}^{-1} \left(\frac{10^4 \text{ cm}^2 \text{ min}}{\text{m}^2} \frac{\text{min}}{60 \text{ s}} \right) = 50,019 \text{ cm}^2 \text{ s}^{-1}$$

Dye studies can also be used to determine first-order reaction rates. For this case the mass under the concentration-time curve is determined at two positions. The rate can then be estimated by

$$k = \frac{1}{\bar{t}_2 - \bar{t}_1} \ln \frac{M_1}{M_2} \quad (10.36)$$

where the travel times and the masses are calculated according to Eqs. 10.32 and 10.31, respectively. We elaborate more on the use of tracers to estimate reaction rates when we discuss the measurement of reaeration later in the text (Lec. 20).

10.5 ESTUARY NUMBER

The relative importance of advection and dispersion can be assessed by a dimensionless analysis of Eq. 10.23:

$$\frac{\partial c}{\partial t} = -U \frac{\partial c}{\partial x} + E \frac{\partial^2 c}{\partial x^2} - kc \quad (10.23)$$

Three dimensionless parameter groups can be defined:

$$c^* = \frac{c}{c_0} \quad (10.37)$$

$$x^* = \frac{kx}{U} \quad (10.38)$$

$$t^* = kt \quad (10.39)$$

These equations can be solved for c , x , and t , respectively, and the results substituted into Eq. 10.23 to yield

$$\frac{\partial c^*}{\partial t^*} = \eta \frac{\partial^2 c^*}{\partial x^{*2}} - \frac{\partial c^*}{\partial x^*} - c^* \quad (10.40)$$

where η is called the *estuary number*,

$$\eta = \frac{kE}{U^2} \quad (10.41)$$

From Eq. 10.41 it should be clear that the magnitude of η determines whether the second derivative term in the differential equation is significant relative to the other terms. Thus the following guidelines can be developed

	Suggested ranges
$\eta \gg 1$	Diffusion predominates
$\eta \approx 1$	Advection/diffusion important
$\eta \ll 1$	Advection predominates
	$\eta > 10$
	$0.1 < \eta < 10$
	$\eta < 0.1$

EXAMPLE 10.5. ESTUARY NUMBER. Evaluate the estuary number for the stream from Example 10.4 for a nonconservative tracer with a half-life of 1 d.

Solution: The parameters must be expressed in common units,

$$E = 50,000 \text{ cm}^2 \text{ s}^{-1} \left(\frac{\text{m}^2}{10,000 \text{ cm}^2} \frac{86,400 \text{ s}}{\text{d}} \right) = 432,000 \text{ m}^2 \text{ d}^{-1}$$

$$U = 24,000 \text{ m d}^{-1}$$

$$k = \frac{0.693}{1 \text{ d}} = 0.693 \text{ d}^{-1}$$

The estuary number can be computed as

$$N = \frac{0.693(432,000)}{(24,000)^2} = 5.2 \times 10^{-4}$$

which clearly indicates a highly advective situation.

PROBLEMS

10.1. A one-dimensional estuary has constant dimensions and flow:

- Width = 1000 ft
- Depth = 10 ft
- Flow = 500 cfs
- Dispersion coefficient = $1 \times 10^6 \text{ m}^2 \text{ d}^{-1}$

(a) Calculate the estuary number for a herbicide that is subject to first-order decay (0.05 d^{-1}) and volatilization (0.3 m d^{-1}).
 (b) Suppose that 10 kg of the herbicide is spilled at mile point 2. Plot the concentration distribution as a function of time at a water intake at mile point 10.

10.2. A tracer study is conducted in a stream with a flow of $3.7 \times 10^5 \text{ m}^3 \text{ d}^{-1}$ and a width of 60 m. At $t = 0$, 50 kg of lithium is instantaneously injected at $x = 0$. Lithium concentration is measured at two downstream stations:

$x = 1 \text{ km}$														$x = 5 \text{ km}$									
t (min)		60	80	100	120	140	160	180	200	220	240	260											
Lithium ($\mu\text{g L}^{-1}$)		0	2	24	78	108	89	52	23	9	3	0											
<hr/>																							
t (min)		550	600	650	700	750	800	850	900	950													
Lithium ($\mu\text{g L}^{-1}$)		0	7	26	47	43	23	8	2	0													

Determine

- (a) velocity, m d^{-1}
- (b) depth, m
- (c) dispersion coefficient, $\text{cm}^2 \text{ s}^{-1}$

10.3. At $t = 0$, 10 g of a conservative algicide is introduced into a trout hatchery pond having $V = 2500 \text{ m}^3$. Flow in the channel and the river is plug flow and the hatchery pond is completely mixed. The dimensions of the channel are $B = 2 \text{ m}$, $H = 1 \text{ m}$, and $L = 0.5 \text{ km}$. Flow in the river upstream of the diversion is $20,000 \text{ m}^3 \text{ d}^{-1}$, and $2000 \text{ m}^3 \text{ d}^{-1}$ is diverted through the hatchery.

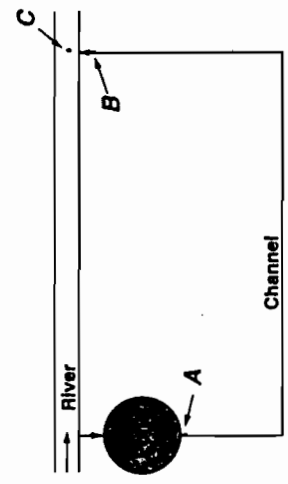


FIGURE P10.3

- (a) Determine the concentration time series at points A, B, and C. Plot your results and express the concentration in ppb.
- (b) How much algicide can be applied if the concentration in the river is never to exceed 10 ppb?

10.4. At $t = 0$, 10 g of a conservative algicide is introduced into a trout hatchery pond having $V = 2500 \text{ m}^3$. Flow in the channel and the river is plug flow and the hatchery pond is completely mixed. The dimensions of the channel are $B = 2 \text{ m}$, $H = 1 \text{ m}$, and $L = 0.5 \text{ km}$. Flow in the river upstream of the diversion is $20,000 \text{ m}^3 \text{ d}^{-1}$, and $2000 \text{ m}^3 \text{ d}^{-1}$ is diverted through the hatchery. Both the river and the pond flow into a well-mixed lake having $V = 10,000 \text{ m}^3$. Determine the time when the maximum concentration occurs in the lake.

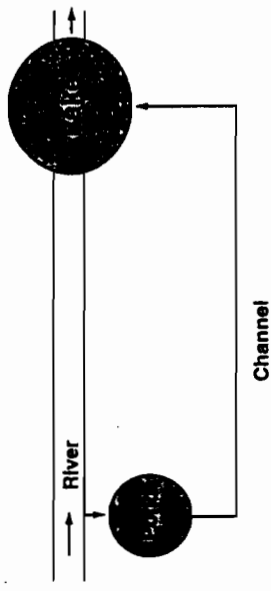


FIGURE P10.4

- 10.5. Suppose that a contaminant is injected into a sediment at a depth of 10 cm. Assuming that the diffusion coefficient is $10^{-6} \text{ cm}^2 \text{ s}^{-1}$.
- (a) How long will it take the contaminant to reach the sediment water interface? Define the front as being the point that encompasses 99% of the contaminant.
- (b) Determine the flux into the overlying water at this time.

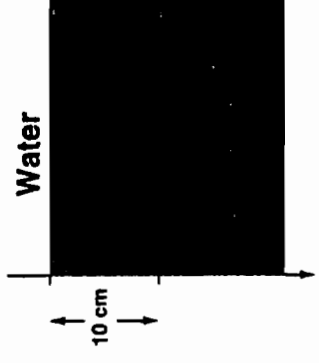


FIGURE P10.5