LECTURE 9

Distributed Systems (Steady-State)

LECTURE OVERVIEW: We now turn to distributed systems. I introduce two idealized reactors that are used to model these systems: plug-flow and mixed-flow reactors. Then, I show how these models can be used to simulate steady-state distributions of pollutants in streams and estuaries.

To this point I have focused on completely mixed reactors or CSTRs. In Lec. 8, I illustrated how incompletely mixed systems can be modeled by breaking them into a coupled set of CSTRs (recall Fig. 8.9). Such characterizations are formally referred to as ***lumped-parameter systems***, so called because each reactor has its own set of parameters. Thus even though the system is continuous, we approximate it as "lumps."

In contrast it is possible to model such systems while maintaining their continuous nature. Such characterizations are formally referred to as ***distributed-parameter systems***. In this lecture I will introduce you to such systems. To do this I focus on two idealized reactors: plug-flow and mixed-flow reactors. I also show how these models can be used to simulate steady-state distributions of pollutants in streams and estuaries.

9.1 IDEAL REACTORS

Both plug-flow and mixed-flow reactors are elongated rectangular basins, as depicted in Fig. 9.1. They are assumed to be well-mixed laterally (y) and vertically (z). Thus we are concerned only with variations in the longitudinal dimension (x).

As in Fig. 9.1, a mass balance is taken for a differential element of length Δx,

$$\Delta V \frac{\partial c}{\partial t} = J_{in} A_c - J_{out} A_c \pm \text{reaction} \tag{9.1}$$

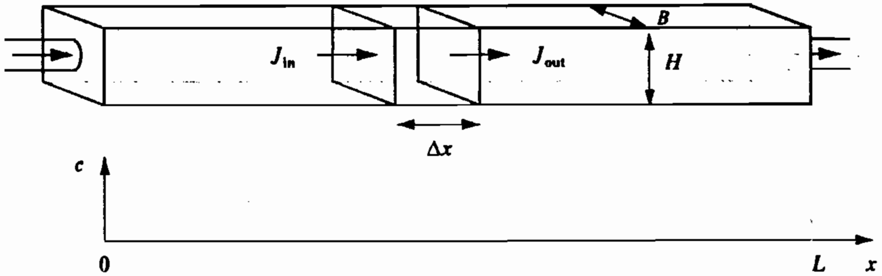


FIGURE 9.1
An elongated rectangular reactor.

where ΔV = volume of the element (L^3) = $A_c \Delta x$
A_c = cross-sectional area of the reactor (L^2) = BH
B = channel width (L)
H = depth (L)
J_{in} and J_{out} = flux of mass in and out of the element due to transport (M L^{-2} T^{-1})
reaction = gain or loss of mass within the element due to reaction (M T^{-1})

9.1.1 Plug-Flow Reactor (PFR)

The plug-flow reactor is one in which advection dominates. As in Fig. 9.2, this means that a "plug" of conservative dye introduced at one end will remain intact as it passes through the reactor. In other words substances are discharged in the same sequence as they enter the reactor.

For a plug-flow reactor, the flux into the element is defined as

$$J_{in} = Uc \tag{9.2}$$

where U = velocity (L T^{-1}) = Q/A_c. The flux out is estimated by a first-order Taylor-series expansion,

$$J_{out} = U\left(c + \frac{\partial c}{\partial x}\Delta x\right) \tag{9.3}$$

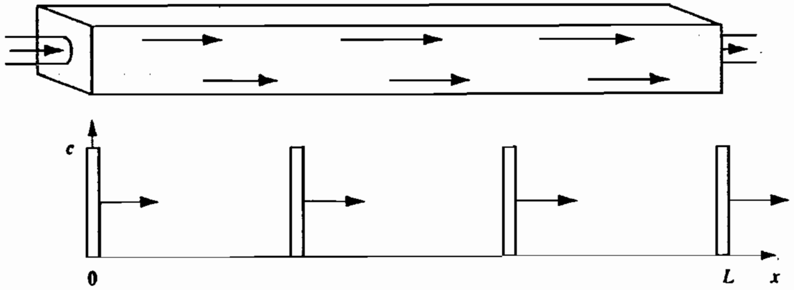


FIGURE 9.2
A plug-flow reactor.

Finally, assuming a first-order decay reaction,

$$\text{Reaction} = -k\Delta V\bar{c} \tag{9.4}$$

where the bar over the c denotes the average value for the element. Substituting the individual terms into Eq. 9.1 yields

$$\Delta V\frac{\partial c}{\partial t} = UA_c c - UA_c\left(c + \frac{\partial c}{\partial x}\,\Delta x\right) - k\,\Delta V\bar{c} \tag{9.5}$$

Combining terms gives

$$\Delta V\frac{\partial c}{\partial t} = -UA_c\frac{\partial c}{\partial x}\,\Delta x - k\,\Delta V\bar{c} \tag{9.6}$$

Dividing by $\Delta V = A_c\,\Delta x$ and taking the limit ($\Delta x \rightarrow 0$) yields

$$\frac{\partial c}{\partial t} = -U\frac{\partial c}{\partial x} - kc \tag{9.7}$$

At steady-state Eq. 9.7 becomes

$$0 = -U\frac{dc}{dx} - kc \tag{9.8}$$

which, if $c = c_0$ at $x = 0$, can be solved for

$$c = c_0 e^{-\frac{k}{U}x} \tag{9.9}$$

EXAMPLE 9.1. PLUG-FLOW REACTOR. In Example 5.2 we used a cascade model to simulate the steady-state distribution of concentration in an elongated tank.

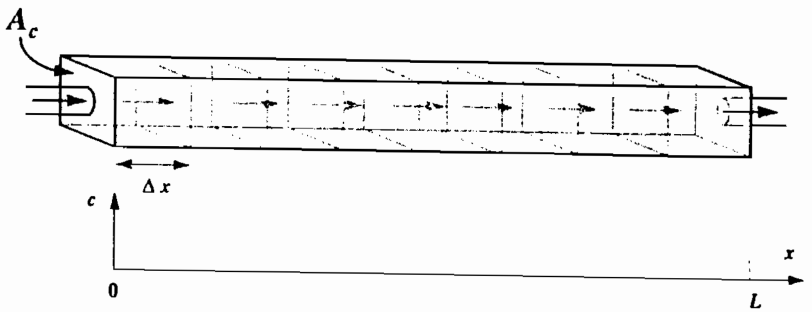


FIGURE E9.1-1

The tank has cross-sectional area $A_c = 10\ \text{m}^2$, length $L = 100$ m, velocity $U = 100\ \text{m hr}^{-1}$, and a first-order reaction rate $k = 2\ \text{hr}^{-1}$. The inflow concentration is 1 mg L^{-1}. Use the plug-flow model to compute the steady-state concentration distribution for the tank. Display the concentrations along with the results for the $n = 4$ and 8 CSTR approximations from Example 5.2.

Solution: The plug-flow model is

$$c = 1e^{-\frac{2}{100}x} = 1e^{-0.02x}$$

Therefore this equation can be used to compute the distribution. The results along with those from Example 5.2 are shown below:

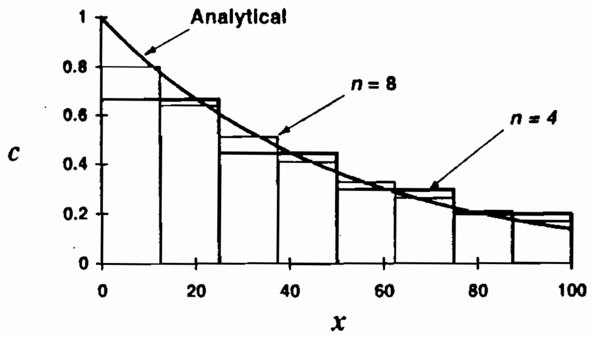


FIGURE E9.1-2

As the number of reactors becomes larger, the cascade model seems to be approaching the analytical solution. In fact it can be proved that the analytical solution is the limit of the cascade approximation.

9.1.2 Comparison of the CSTR and the PFR

Performance of the CSTR and the plug-flow reactor can now be compared. A nice way to do this is via the residence time required to achieve a certain level of performance efficiency. For example recall that the steady-state solution for the CSTR with first-order reaction can be represented as

$$c = c_{in}\frac{Q}{Q + kV} \tag{9.10}$$

The performance efficiency can be defined in terms of the transfer function (Eq. 3.22)

$$\beta \equiv \frac{c}{c_{in}} = \frac{Q}{Q + kV} \tag{9.11}$$

Dividing the numerator and the denominator by Q gives

$$\beta = \frac{1}{1 + k\tau_w} \tag{9.12}$$

where τ_w = water residence time = V/Q. This result can be solved for residence time,

$$\tau_w = \frac{1}{k}\frac{1 - \beta}{\beta} \tag{9.13}$$

In a similar fashion the efficiency of the PFR can be defined as

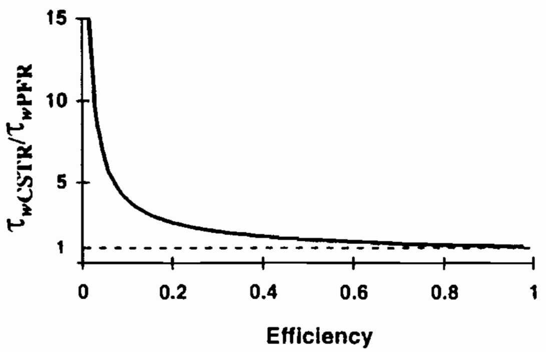


FIGURE 9.3
Ratio of CSTR to PFR residence times versus efficiency.

$$\beta \equiv \frac{c}{c_{in}} = e^{-\frac{k}{U}x} \tag{9.14}$$

Recognizing that the residence time of the PFR is equal to L/U,

$$\beta = e^{-k\tau_w} \tag{9.15}$$

This result can also be solved for residence time,

$$\tau_w = \frac{1}{k}\ln\left(\frac{1}{\beta}\right) \tag{9.16}$$

The two types of reactors can be compared by plotting the ratio of the two residence times versus efficiency. The result is shown in Fig. 9.3. This plot shows that the CSTR always requires a higher residence time to achieve a desired efficiency. This leads to the general conclusion that for first-order reactions the PFR is more efficient than the CSTR for removing pollutants.

9.1.3 Mixed-Flow Reactor (MFR)

The mixed-flow reactor is one in which both advection and diffusion/dispersion are important. As in Fig. 9.4, this means that a plug of conservative dye introduced at one end will "spread out" as it passes through the reactor.

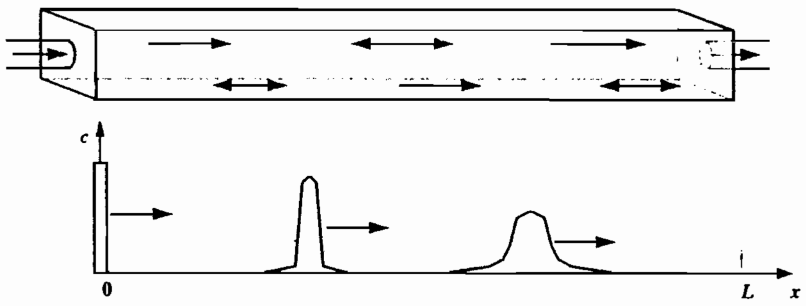


FIGURE 9.4
A mixed-flow reactor.

For the mixed-flow reactor, the flux in is defined as

$$J_{in} = Uc - E\frac{\partial c}{\partial x} \tag{9.17}$$

where E = turbulent diffusion. Note that the second term is Fick's first law (Eq. 8.2). The flux out is

$$J_{out} = U\left(c + \frac{\partial c}{\partial x}\Delta x\right) - E\left[\frac{\partial c}{\partial x} + \frac{\partial}{\partial x}\left(\frac{\partial c}{\partial x}\right)\Delta x\right] \tag{9.18}$$

Substituting the individual terms into Eq. 9.1 yields

$$\Delta V\frac{\partial c}{\partial t} = UA_c c - UA_c\left(c + \frac{\partial c}{\partial x}\Delta x\right)$$
$$- EA_c\frac{\partial c}{\partial x} + EA_c\left[\frac{\partial c}{\partial x} + \frac{\partial}{\partial x}\left(\frac{\partial c}{\partial x}\right)\Delta x\right] - k\,\Delta V\bar{c} \tag{9.19}$$

Combining terms gives

$$\Delta V\frac{\partial c}{\partial t} = -UA_c\frac{\partial c}{\partial x}\Delta x + EA_c\frac{\partial^2 c}{\partial x^2}\Delta x - k\,\Delta V\bar{c} \tag{9.20}$$

Dividing by $\Delta V = A_c\,\Delta x$ and taking the limit ($\Delta x \to 0$) yields

$$\frac{\partial c}{\partial t} = -U\frac{\partial c}{\partial x} + E\frac{\partial^2 c}{\partial x^2} - kc \tag{9.21}$$

or at steady-state,

$$0 = -U\frac{dc}{dx} + E\frac{d^2 c}{dx^2} - kc \tag{9.22}$$

The general solution can be obtained in a variety of ways. A simple approach is to assume that the solution has the form

$$c = e^{\lambda x} \tag{9.23}$$

This solution can be substituted into Eq 9.22 to arrive at the characteristic equation

$$E\lambda^2 - U\lambda - k = 0 \tag{9.24}$$

which can be solved for

$$\begin{matrix}\lambda_1 \\ \lambda_2\end{matrix} = \frac{U}{2E}\left(1 \pm \sqrt{1 + \frac{4kE}{U^2}}\right) = \frac{U}{2E}(1 \pm \sqrt{1 + 4\eta}) \tag{9.25}$$

where $\eta = kE/U^2$.

Therefore the general solution is

$$c = Fe^{\lambda_1 x} + Ge^{\lambda_2 x} \tag{9.26}$$

where F and G = constants of integration.

The constants of integration can be evaluated by boundary conditions. For the tank in Fig. 9.1 a boundary condition can be developed by taking a mass balance at the inlet,

$$Qc_{in} = Qc(0) - EA_c\frac{dc}{dx}(0) \qquad (9.27)$$

Dividing by A_c and substituting Eq. 9.26 gives

$$(U - E\lambda_1)F + (U - E\lambda_2)G = Uc_{in} \qquad (9.28)$$

The second boundary condition relates to the fact that no diffusion of mass is assumed to occur through the outlet. Therefore no gradient should exist at the end of the tank,

$$\frac{dc}{dx}(L) = 0 \qquad (9.29)$$

Using Eq. 9.26,

$$\left(\lambda_1 e^{\lambda_1 L}\right)F + \left(\lambda_2 e^{\lambda_2 L}\right)G = 0 \qquad (9.30)$$

These two boundary conditions are commonly called ***Danckwerts boundary conditions***, after the chemical engineer P. V. Danckwerts, who proposed them originally (Danckwerts 1953).

Equations 9.28 and 9.30 now represent a system of two equations with two unknowns. They can be solved for

$$F = \frac{Uc_{in}\lambda_2 e^{\lambda_2 L}}{(U - E\lambda_1)\lambda_2 e^{\lambda_2 L} - (U - E\lambda_2)\lambda_1 e^{\lambda_1 L}} \qquad (9.31)$$

and

$$G = \frac{Uc_{in}\lambda_1 e^{\lambda_1 L}}{(U - E\lambda_2)\lambda_1 e^{\lambda_1 L} - (U - E\lambda_1)\lambda_2 e^{\lambda_2 L}} \qquad (9.32)$$

These constants can be substituted into Eq. 9.26, which can then be used to calculate concentration along the tank's length.

EXAMPLE 9.2. MIXED-FLOW REACTOR. For the same system as in Example 9.1, compute concentration using the mixed-flow model with diffusion coefficients of E = 2000 and 10,000 m^2 hr^{-1}. Plot your results along with the PFR and CSTR models for the same tank.

Solution: The following results can be generated:

	E = 0(PFR)	E = 2000	E = 10,000	E = ∞(CSTR)
λ_1	∞	0.0653	0.02	0.0
λ_2	−0.02	−0.0153	−0.01	0.0
F	0	5.66×10^{-5}	0.0126	0.1667
G	1	0.7656	0.5063	0.1667
x = 0	1.0000	0.7656	0.5189	0.3333
x = 20	0.6703	0.5638	0.4333	0.3333
x = 40	0.4493	0.4157	0.3674	0.3333
x = 60	0.3012	0.3083	0.3197	0.3333
x = 80	0.2019	0.2354	0.2899	0.3333
x = 100	0.1353	0.2044	0.2794	0.3333

They can be displayed as

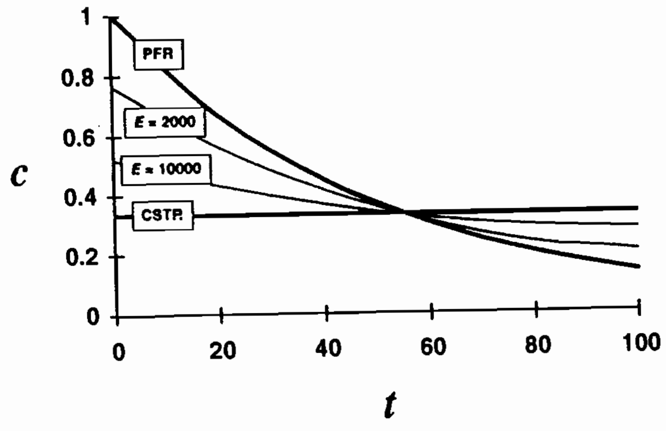


FIGURE E9.2

Thus at zero diffusion, the model converges on the PFR. As diffusion is increased it approaches the CSTR.

BOX 9.1. The Peclet Number

A dimensionless analysis of Eq. 9.22 can be used to gain insight into the results of Example 9.2. To do this, the following dimensionless parameter groups can be defined:

$$x^* = \frac{x}{L} \qquad (9.33)$$

$$c^* = \frac{c}{c_{in}} \qquad (9.34)$$

These can be solved for x and c and the result substituted into Eq. 9.22 to give

$$0 = -\frac{U}{L}\frac{dc^*}{dx^*} + \frac{E}{L^2}\frac{d^2c^*}{dx^{*2}} - kc^* \qquad (9.35)$$

Multiplying this equation by L/U yields

$$0 = -\frac{dc^*}{dx^*} + \frac{E}{UL}\frac{d^2c^*}{dx^{*2}} - \frac{kL}{U}c^* \qquad (9.36)$$

or

$$0 = -\frac{dc^*}{dx^*} + \frac{1}{P_e}\frac{d^2c^*}{dx^{*2}} - D_a c^* \qquad (9.37)$$

where P_e is called the ***Peclet number***,

$$P_e = \frac{LU}{E} = \frac{\text{rate of advective transport}}{\text{rate of diffusive/dispersive transport}} \qquad (9.38)$$

and D_a is called the ***Damkohler number***,

$$D_a = \frac{kL}{U} = \frac{\text{rate of consumption by decay}}{\text{rate of advective transport}} \tag{9.39}$$

Thus if P_e is high (> 10), the system approaches a PFR because the first-derivative term dominates the second-derivative term in Eq. 9.37. Conversely for low $P_e(< 0.1)$, longitudinal mixing dominates and the system becomes more like a CSTR. For intermediate values the MFR model is required.

9.2 APPLICATION OF THE PFR MODEL TO STREAMS

Just as the CSTR is the fundamental model for a lake, the PFR is the fundamental model for a stream. In this section we emphasize how a plug-flow model can be applied to analyze two types of sources: point and distributed loadings (Fig. 9.5).

9.2.1 Point Source

The following analysis focuses on the case where a point-source loading is injected into a channel having constant characteristics. For this case the solution would follow Eq. 9.9.

The initial concentration c_0 can be computed by taking a mass balance at the injection point. If it is assumed that complete mixing takes place in the lateral and vertical dimensions, the situation is as depicted in Fig. 9.6. A flow balance yields

$$Q = Q_w + Q_r \tag{9.40}$$

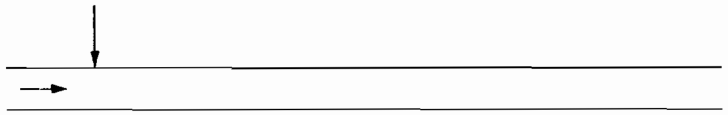
A mass balance can then be developed,

$$0 = Q_w c_w + Q_r c_r - (Q_w + Q_r)c_0 \tag{9.41}$$

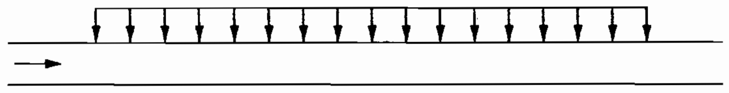
that can be solved for

$$c_0 = \frac{Q_w c_w + Q_r c_r}{Q_w + Q_r} \tag{9.42}$$

or (because $Q_w c_w = W$)



(a) Point source



(b) Nonpoint or distributed source

FIGURE 9.5
Point and nonpoint sources for one-dimensional systems.

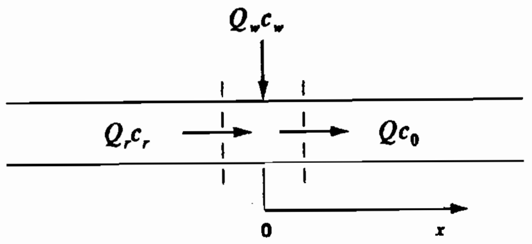


FIGURE 9.6
Mass balance for a point source discharged into a plug-flow system.

$$c_0 = \frac{W + Q_r c_r}{Q_w + Q_r} \tag{9.43}$$

Now, in some cases, the waste source has a relatively small flow and a high concentration, whereas the river has a relatively high flow and a small concentration. In other words,

$$Q_r \gg Q_w \tag{9.44}$$

and

$$c_r \ll c_w \tag{9.45}$$

For these cases, the following approximation holds:

$$c_0 = \frac{W}{Q} \tag{9.46}$$

EXAMPLE 9.3. POINT SOURCE FOR A PLUG-FLOW SYSTEM. A point source is discharged to a river having the following characteristics: $Q_r = 12 \times 10^6$ m^3 d^{-1}, $c_r = 1$ mg L^{-1}, $Q_w = 0.5 \times 10^6$ m^3 d^{-1}, and $c_w = 400$ mg L^{-1}.

(*a*) Determine the initial concentration assuming complete mixing vertically and laterally. Assess whether Eq. 9.46 is a good approximation for this case.

(*b*) Calculate the concentration of the pollutant for 8 km below the injection point. Note that the stream has a cross-sectional area of 2000 m^2 and the pollutant reacts with first-order decay ($k = 0.8$ d^{-1}).

Solution: (*a*) First, Eq. 9.42 can be used to compute the initial concentration,

$$c_0 = \frac{0.5 \times 10^6(400) + 12 \times 10^6(1)}{0.5 \times 10^6 + 12 \times 10^6} = 16.96 \text{ mg L}^{-1}$$

Equation 9.46 can also be used to make the same estimate, with the result

$$c_0 = \frac{0.5 \times 10^6(400)}{0.5 \times 10^6 + 12 \times 10^6} = 16 \text{ mg L}^{-1}$$

which represents a 5.7% error. This might be considered too large an error and the approximation deemed inadequate.

(*b*) The velocity in the stream can be computed as

$$U = \frac{Q}{A_c} = \frac{12.5 \times 10^6}{2000} = 6250 \text{ m d}^{-1}$$

Using Eq. 9.9,

$$c = 16.96e^{-\frac{0.8}{6250}x}$$

Substituting values of x gives

x (km)	0	1.6	3.2	4.8	6.4	8
c (mg L^{-1})	16.96	13.82	11.26	9.18	7.48	6.09

Thus the pollutant is decaying exponentially as it is being transported downstream.

9.2.2 Distributed Source

A ***distributed source*** is one that enters a water body in a diffuse manner.[†] In the context of a channel, it would enter along its length. The simplest type is the uniform distributed source, which is constant spatially (Fig. 9.7).

The distributed source can be incorporated into the steady-state mass balance,

$$0 = -U\frac{dc}{dx} - kc + S_d \tag{9.47}$$

If $c = c_0$ at $x = 0$, then

$$c = c_0 e^{-\frac{k}{U}x} + \frac{S_d}{k}\left(1 - e^{-\frac{k}{U}x}\right) \tag{9.48}$$

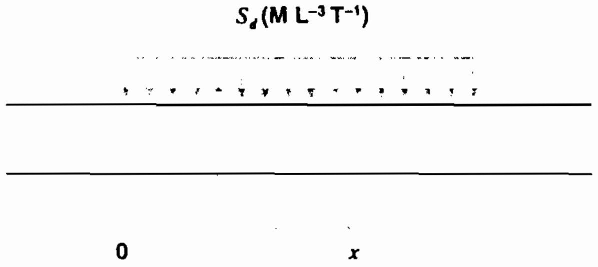


FIGURE 9.7
Uniform distributed source.

EXAMPLE 9.4. DISTRIBUTED SOURCE INTO A PLUG-FLOW SYSTEM. Suppose that for the same problem setting as in Example 9.3, a uniform diffuse source load with no flow contribution begins 8 km downstream from the point source. The diffuse source has a value of 15 g m^{-3} d^{-1} and continues for an additional 8 km. Thereafter all loadings terminate. Determine the concentration for the stretch 24 km below the point source. Note that the area increases to 3000 m^2 at the 8-km point.

Solution: The evaluation from Example 9.3 holds for the first 8 km and provides the boundary condition for the next reach. For km 8 to km 16 the velocity must be recalculated because of the increased area,

$$U = \frac{12.5 \times 10^6}{3000} = 4167 \text{ m d}^{-1}$$

Equation 9.48 can then be used to calculate the concentration in the second 8-km stretch,

$$c = 6.09e^{-\frac{0.8}{4167}(x-8000)} + \frac{15}{0.8}\left(1 - e^{-\frac{0.8}{4167}(x-8000)}\right)$$

Substituting values of x gives

x (km)	8	9.6	11.2	12.8	14.4	16
c (mg L^{-1})	6.09	9.44	11.90	13.71	15.05	16.03

Finally Eq. 9.9 can then be used to calculate the concentration in the third 8-km stretch,

$$c = 16.03e^{-\frac{0.8}{4167}(x-16{,}000)}$$

Substituting values of x gives

x (km)	16	17.6	19.2	20.8	22.6	24
c (mg L^{-1})	16.03	11.79	8.67	6.38	4.69	3.45

The entire solution can be displayed graphically as

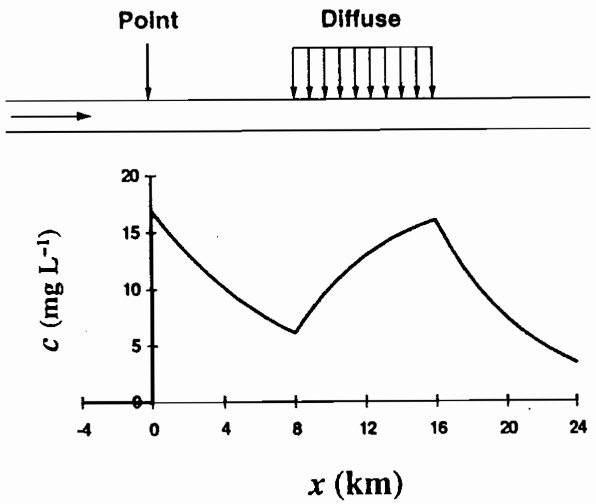


FIGURE E9.4

[†] Note that the distributed sources described here contribute mass but no flow. In Lec. 22 we will model distributed sources that contribute both mass and flow.

9.3 APPLICATION OF THE MFR MODEL TO ESTUARIES

The mixed-flow system is the fundamental model for one-dimensional estuaries. In this section I emphasize how the mixed-flow model can be applied to analyze point and distributed loadings.

9.3.1 Point Source

The following analysis focuses on the case where a point-source loading is injected into a channel having constant characteristics. The impact of dispersion on the initial concentration at a point source can be computed by taking a mass balance at the injection point. If it is assumed that complete mixing takes place in the lateral and vertical dimensions, the situation is as depicted in Fig. 9.8. For simplicity we assume that the loading does not add significant flow at the mixing point (that is, relative to the flow in the estuary). Problem 9.8 explores the case where the waste flow significantly elevates the estuary flow.

We use the same approach as was used to solve Eq. 9.22, and the general solution for this case is

$$c = Fe^{\lambda_1 x} + Ge^{\lambda_2 x} \tag{9.26}$$

where the λ's are

$$\begin{matrix}\lambda_1 \\ \lambda_2\end{matrix} = \frac{U}{2E}\left(1 \pm \sqrt{1 + 4\eta}\right) \tag{9.25}$$

The constants of integration can be evaluated by boundary conditions. For the present case these are

$$c = 0 \qquad @\, x = -\infty \tag{9.49}$$

$$c = 0 \qquad @\, x = \infty \tag{9.50}$$

Applying these boundary conditions reduces the number of unknowns to one,

$$c_1 = c_0 e^{\lambda_1 x} \qquad \text{for } x \le 0 \tag{9.51}$$

$$c_2 = c_0 e^{\lambda_2 x} \qquad \text{for } x > 0 \tag{9.52}$$

The mass balance depicted in Fig. 9.8 can now be used to evaluate c_0. For the case where the waste load contributes negligible flow, the mass balance is

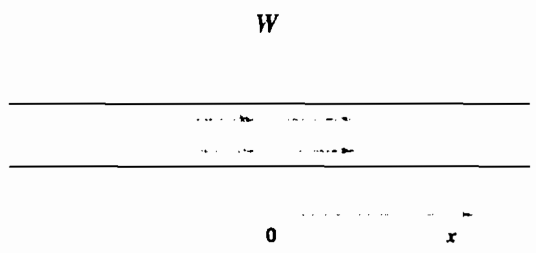


FIGURE 9.8
Mass balance for a point source discharged into a mixed-flow system.

$$W + UAc_1(0) - EA\frac{dc_1}{dx}(0) - UAc_2(0) + EA\frac{dc_2}{dx}(0) = 0 \tag{9.53}$$

Substituting Eqs. 9.51 and 9.52 yields

$$W + UAc_0 - EA\lambda_1 c_0 - UAc_0 + EA\lambda_2 c_0 = 0 \tag{9.54}$$

which can be solved for

$$c_0 = \frac{W}{Q}\frac{1}{\sqrt{1 + 4\eta}} \tag{9.55}$$

Therefore the final solution is

$$c = \frac{W}{Q\sqrt{1 + 4\eta}}e^{\frac{U}{2E}\left(1 + \sqrt{1+4\eta}\right)x} \qquad x \le 0 \tag{9.56}$$

$$c = \frac{W}{Q\sqrt{1 + 4\eta}}e^{\frac{U}{2E}\left(1 - \sqrt{1+4\eta}\right)x} \qquad x \ge 0 \tag{9.57}$$

Note that when E approaches zero, this model converges on the plug-flow model (see Prob. 9.9). Also observe that Eq. 9.55 is in the format of Eq. 1.8. That is, it relates a critical concentration c_0 to the loading W. Thus as illustrated in the following example, it provides a means to perform an assimilative capacity calculation for an estuary.

EXAMPLE 9.5. POINT SOURCE TO A MIXED-FLOW SYSTEM. A point source is discharged into an estuary having the following characteristics:

	Value	Units
Dispersion coefficient	80×10^6	$m^2\, d^{-1}$
Flow	12×10^3	$m^3\, d^{-1}$
Width	0.5	m
Depth	8	m

The pollutant settles ($v = 1$ m d^{-1}) and reacts via first-order kinetics ($k = 0.2$ d^{-1}). What mass loading could be input to this system under steady-state conditions if the allowable concentration at the outfall is 10 ppm? Express your results in kg yr^{-1}. Assume complete lateral and vertical mixing at the outfall.

Solution: The velocity can be calculated,

$$U = \frac{12 \times 10^3}{0.5 \times 8} = 3000 \text{ m d}^{-1}$$

Equation 9.55 can be used to compute

$$W = Qc_0\sqrt{1 + \frac{4\left(k + \frac{v}{H}\right)E}{U^2}}$$

$$= 12 \times 10^3(10)\sqrt{1 + \frac{4\left(0.2 + \frac{1}{8}\right)80 \times 10^6}{(3000)^2}} = 425{,}206 \text{ g d}^{-1}$$

or converting to the proper units,

$$W = 425{,}206 \text{ g d}^{-1}\left(\frac{365 \text{ d}}{\text{yr}}\,\frac{\text{kg}}{10^3 \text{g}}\right) = 155{,}200 \text{ kg yr}^{-1}$$

9.3.2 Distributed Source

Thomann and Mueller (1987) present the following solution for the case of a distributed load of a first-order decaying substance into an advective/dispersive system (see Fig. 9.9):

$$c = \frac{S_d}{k}\left(\frac{\sqrt{1 + 4\eta} - 1}{2\sqrt{1 + 4\eta}}\right)\left(1 - e^{-\frac{U}{2E}\left(1+\sqrt{1+4\eta}\right)a}\right)e^{\frac{U}{2E}\left(1+\sqrt{1+4\eta}\right)x} \qquad x \leq 0 \qquad (9.58)$$

$$c = \frac{S_d}{k}\left[1 - \frac{\sqrt{1 + 4\eta} - 1}{2\sqrt{1 + 4\eta}}e^{\frac{U}{2E}\left(1+\sqrt{1+4\eta}\right)(x-a)} - \frac{\sqrt{1 + 4\eta} + 1}{2\sqrt{1 + 4\eta}}e^{\frac{U}{2E}\left(1-\sqrt{1+4\eta}\right)x}\right] \qquad 0 \leq x \leq a \qquad (9.59)$$

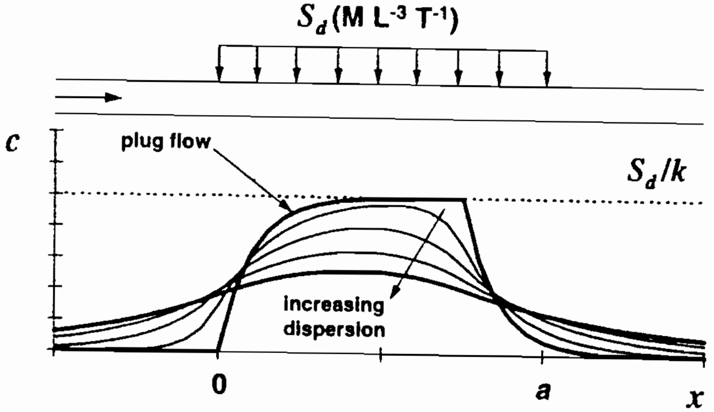


FIGURE 9.9
Uniform distributed source in a mixed-flow system. Several different levels of dispersion are displayed along with the plug-flow solution.

$$c = \frac{S_d}{k}\left(\frac{\sqrt{1 + 4\eta} + 1}{2\sqrt{1 + 4\eta}}\right)\left(1 - e^{\frac{U}{2E}\left(1-\sqrt{1+4\eta}\right)a}\right)e^{\frac{U}{2E}\left(1-\sqrt{1+4\eta}\right)(x-a)} \qquad x \geq a \qquad (9.60)$$

As depicted in Fig. 9.9 the solution approaches the plug-flow case when *E* is low. As *E* increases, the solution eventually approaches a bell-shaped distribution centered on the midpoint of the distributed loading.

PROBLEMS

9.1. For the mixed-flow model notice that $c(0) \leq c_{in}$. That is, the concentration immediately inside the reactor will be less than or equal to the inflow concentration (Fig. E9.2). When will they be equal? Use mathematical and physical arguments to establish why they are different.

9.2. Determine the Peclet numbers for the four cases represented in Example 9.2.

9.3. At the end of Box 9.1 we specified bounds for the Peclet number. Verify these bounds by determining concentration profiles for the tank from Example 9.2 using Peclet numbers of 0.1 and 10 (by changing *E*). Display your results graphically along with the CSTR and PFR solutions.

9.4. Determine the steady-state distribution of a pollutant ($k = 0.1$ d^{-1}) for $x = 0$ to 32 km in the following system. Note that the concentration in the river immediately upstream of the distributed load is 5 mg L^{-1}. How far downstream will the system return to the level of 5 mg L^{-1}?

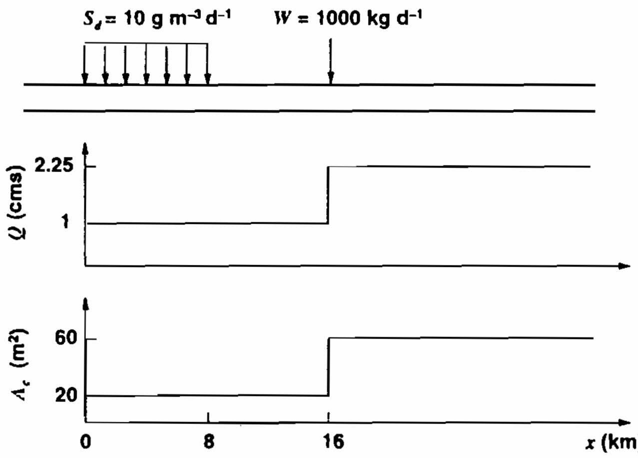


FIGURE P9.4

9.5. A point source is discharged into an estuary having the following characteristics:

	Value	Units
Dispersion coefficient	10^5	cm^2 s^{-1}
Flow	5×10^4	m^3 d^{-1}
Width	100	m
Depth	2	m